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Problem Sheet 2

C3.4: Algebraic Geometry

13 November, 2021

Section A: Introductory

Question 1. Projective space is locally affine

For $0 \leqslant j \leqslant n$, consider the map $\phi_j : U_j \to \mathbb{A}^n$ defined in lectures, mapping the j th affine coordinate patch $U_j \subseteq \mathbb{P}^n$ to affine n-space. Using the Zariski topology of projective varieties on $\mathbb{P}^n$ restricted to U_j, and the Zariski topology of affine varieties on $\mathbb{A}^n$, show that ϕ_j is a homeomorphism of topological spaces.

Proof. Without loss generality we fix $j = 0$. Then $\varphi_0 : U_0 \to \mathbb{A}^n$ is given by $[x_0 : x_1 : \cdots : x_n] \mapsto \left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$. It has an inverse $\varphi_0^{-1} : \mathbb{A}^n \to U_0$ given by $(x_1, \ldots, x_n) \mapsto [1 : x_1 : \cdots : x_n]$. To show that φ_0 is a homeomorphism, it suffices to show that

$$X \text{ is closed in } U_0 \subseteq \mathbb{P}^n \iff \varphi_0(X) \text{ is closed in } \mathbb{A}^n$$

$\implies$ Suppose that X is closed in U_0. Then $X = \mathbb{V}(\langle f_1, \ldots, f_m\rangle) \cap U_0$, where $f_1, \ldots, f_m \in k[x_0, \ldots, x_n]$ are homogeneous polynomials. Let $\pi\colon k[x_0, \ldots, x_n] \to k[x_1, \ldots, x_n]$ be the ring homomorphism induced by $x_0 \mapsto 1$. By definition we have $\pi(f_i) = f_i \circ \varphi_0^{-1}$ as maps on $\mathbb{A}^n$. Then for $a \in \mathbb{A}^n$,

$$\begin{aligned} a \in \varphi_0(X) &\iff \varphi_0^{-1}(a) \in X \\ &\iff \forall\, i \in \{1, \ldots, m\}\ f_i \circ \varphi_0^{-1}(a) = 0 \\ &\iff \forall\, i \in \{1, \ldots, m\}\ \pi(f_i)(a) = 0 \\ &\iff a \in \mathbb{V}(\langle \pi(f_1), \ldots, \pi(f_m)\rangle) \end{aligned}$$

Hence $\varphi_0(X) = \mathbb{V}(\langle \pi(f_1), \ldots, \pi(f_m)\rangle)$ is closed in $\mathbb{A}^n$.

$\impliedby$ Suppose that $\varphi_0(X)$ is closed in $\mathbb{A}^n$. Let $\varphi_0(X) = \mathbb{V}(\langle g_1, \ldots, g_k\rangle)$. Let $\iota\colon k[x_1, \ldots, x_n] \to k[x_0, x_1, \ldots, x_n]$ be the homogenisation. Then by definition $\iota(g_i) = g_i \circ \varphi_0$ as maps on U_0. For $a \in U_0$,

$$\begin{aligned} a \in X &\iff \varphi_0(a) \in \varphi_0(X) \\ &\iff \forall\, i \in \{1, \ldots, k\}\ g_i \circ \varphi_0(a) = 0 \\ &\iff \forall\, i \in \{1, \ldots, k\}\ \iota(g_i)(a) = 0 \\ &\iff a \in \mathbb{V}(\langle \iota(g_1), \ldots, \pi(g_k)\rangle) \end{aligned}$$

Hence $X = U_0 \cap \mathbb{V}(\langle \iota(g_1), \ldots, \pi(g_k)\rangle)$ is closed in U_0.

We conclude that φ_0 is a homeomorphism. $\square$

Question 2. Projective closures and affine cones

(a) Let X be the parabola $\mathbb{V}(y - x^2) \subseteq \mathbb{A}^2$. What is its projective closure $\overline{X} \subseteq \mathbb{P}^2$? Draw the affine cone $\widehat{\overline{X}}$ over $\overline{X}$, in $\mathbb{A}^3$, and identify the line corresponding to the "point at infinity" on $\overline{X}$.

(b) Show that the affine varieties $\mathbb{V}(y - x^2) \subseteq \mathbb{A}^2$ and $\mathbb{V}(y - x^3) \subseteq \mathbb{A}^2$ are isomorphic. Can you give an intuitive explanation why their two projective closures in $\mathbb{P}^2$ are not projectively equivalent (isomorphic)? We do not have the tools yet to prove that they are non-isomorphic; try and find an intuitive reason.

Proof. (a) The projective closure is obtained by taking the homogenisations of the generators of the vanishing ideal. Then $\overline{X} = \mathbb{V}(yz - x^2) \subseteq \mathbb{P}^2$. The affine cone of $\overline{X}$ in $\mathbb{A}^3$ is shown in the following diagram:

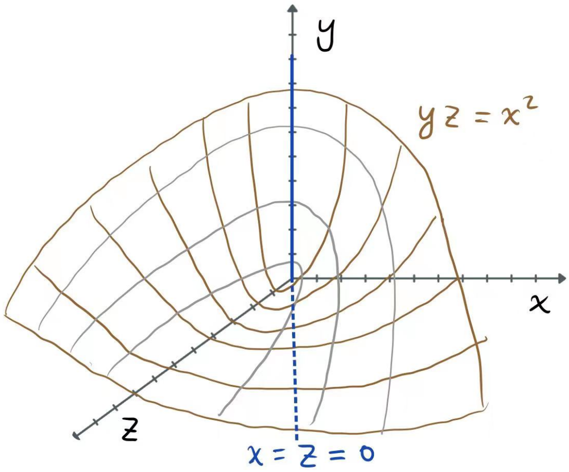

Figure 1: $\mathbb{V}(yz-x^2)$ in $\mathbb{R}^3$ for $y, z > 0$. The variety also has a symmetric part in $y, z < 0$.

where the line of infinity corresponds to $x = z = 0$.

(b) Let $\varphi\colon \mathbb{A}^1 \to \mathbb{V}(y-x^2)$ be a morphism given by $t \mapsto (t, t^2)$. Then it is easy to see that φ is an isomorphism. Similarly $\psi\colon \mathbb{A}^1 \to \mathbb{V}(y-x^3)$, $t \mapsto (t, t^3)$ is also an isomorphism. Hence we have $\mathbb{V}(y-x^2) \cong \mathbb{V}(y-x^3)$ as affine varieties.

Their projection closures are given by $\mathbb{V}(yz-x^2)$ and $\mathbb{V}(yz^2-x^3) \subseteq \mathbb{P}^2$. We have $\deg \mathbb{V}(yz-x^2) = 2 \neq 3 = \deg \mathbb{V}(yz^2-x^3)$. Note that a projective linear transformation does not change the degree of the variety because it maps a projective line to a projective line in $\mathbb{P}^2$. Therefore the two projective varieties cannot be projectively equivalent. $\square$

Section B: Core

Question 3. The Twisted Cubic

The projective variety $C = \mathbb{V}(F_0, F_1, F_2) \subseteq \mathbb{P}^3$, where

$$F_0(z_0, z_1, z_2, z_3) = z_0 z_2 - z_1^2$$
$$F_1(z_0, z_1, z_2, z_3) = z_0 z_3 - z_1 z_2$$
$$F_2(z_0, z_1, z_2, z_3) = z_1 z_3 - z_2^2$$

is known as the *twisted cubic*.

(a) Show that C is equal to the image of the Veronese map

$$\nu : \mathbb{P}^1 \to \mathbb{P}^3$$
$$\nu : [x_0 : x_1] \mapsto [x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3]$$

(b) Restrict to the affine patch $U_0 \subseteq \mathbb{P}^3$ given by setting $z_0 = 1$. Show that $C \cap U_0$ is equal to $\mathbb{V}(f_0, f_1) \subseteq \mathbb{A}^3$, where $f_i(z_1, z_2, z_3) := F_i(1, z_1, z_2, z_3)$ for $i = 1, 2$.

(c) For $i = 0, 1, 2$ we write Q_i for the quadric hypersurface $\mathbb{V}(F_i) \subseteq \mathbb{P}^3$. Show that, for $i \neq j$, the hypersurfaces Q_i and Q_j intersect in the union of C and a line. Therefore no two of them alone may be used to define C. Deduce that the homogenizations of the generators of an affine ideal do not necessarily generate the homogeneous ideal of the projective closure, showing that indeed we need to homogenise all elements of the affine ideal.

Proof. (a) We can directly verify that $\operatorname{im} \nu \subseteq C$:

$$F_0\left(x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3\right) = x_0^3 \cdot x_0 x_1^2 - (x_0^2 x_1)^2 = 0$$
$$F_1\left(x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3\right) = x_0^3 x_1^3 - x_0^2 x_1 \cdot x_0 x_1^2 = 0$$
$$F_2\left(x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3\right) = x_0^2 x_1 \cdot x_1^3 - (x_0 x_1^2)^2 = 0$$

On the other hand, suppose that $[z_0 : z_1 : z_2 : z_3] \in C$. Since $t \mapsto t^3$ is surjective in k, we can put $z_0 = t^3$ and $z_3 = s^3$. Then

$$t^3 z_2 = z_1^2, \qquad t^3 s^3 = z_1 z_2, \qquad s^3 z_1 = z_2^2$$

From the first two equations, we have

$$z_1^3 = t^3 z_1 z_2 = t^6 s^3 \implies z_1 = \omega t^2 s$$

where ω is a third root of unity. Let $u = \omega s$. Then the third equation implies that

$$u^4 t^2 = z_2^2 \implies z_2 = \pm u^2 t$$

substituting into the second equation we obtain that $z_2 = u^2 t$. Hence $[z_0 : z_1 : z_2 : z_3] = [t^3 : t^2 u : t u^2 : u^3] \in \operatorname{im} \nu$.

We conclude that $\operatorname{im} \nu = C$.

(b) We have $C \cap U_0 = \left\{(x_1, x_1^2, x_1^3) : x_1 \in \mathbb{A}^1\right\}$ and $\mathbb{V}\left(f_0, f_1\right) = \mathbb{V}\left(z_2 - z_1^2, z_3 - z_1 z_2\right) \subseteq U_0$. It is obvious that $C \cap U_0 = \mathbb{V}\left(f_0, f_1\right)$ in $U_0 \cong \mathbb{A}^3$.

(c) By observation

Why? Some justification is needed!

$$Q_0 \cap Q_1 = \mathbb{V}(F_0, F_1) = C \cup \{z_0 = z_1 = 0\}$$
$$Q_0 \cap Q_2 = \mathbb{V}(F_0, F_2) = C \cup \{z_1 = z_2 = 0\}$$
$$Q_1 \cap Q_2 = \mathbb{V}(F_1, F_2) = C \cup \{z_2 = z_3 = 0\}$$

This shows that only homogenising the generators of the affine ideal is not sufficient. □

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Question 4. Veronese varieties

(a) Show that any projective variety is isomorphic to the intersection of a Veronese variety with a linear space, the projectivisation $\mathbb{P}(V)$ of some k-vector subspace $V \subseteq k^{n+1}$.

(b) Deduce that any projective variety is isomorphic to an intersection of quadric hypersurfaces.

Proof. (a) Let $X = \mathbb{V}(I) \subseteq \mathbb{P}^n$ be a projective variety where I is a finitely generated homogeneous ideal. We claim that X can be the of vanishing loci of homogeneous polynomials *of the same degree*. Suppose that $I = \langle f_1, \ldots, f_k \rangle$ with $\deg f_i = d_i$. Let $d := \max\{d_i : 1 \leqslant i \leqslant k\}$. For each i, we note that

$$\mathbb{V}(f_i) = \mathbb{V}\left(x_0^{d-d_i} f_i, \ldots, x_n^{d-d_i} f_i\right)$$

Hence

$$X = \bigcap_{i=1}^{k} \mathbb{V}(f_i) = \bigcap_{i=1}^{k} \mathbb{V}\left(x_0^{d-d_i} f_i, \ldots, x_n^{d-d_i} f_i\right) = \mathbb{V}(J)$$

where J is generated by homogeneous polynomials of degree d. For simplicity we rewrite the generators for J as $g_1, \ldots, g_\ell$, where $g_i = \sum_{|I|=d} a_{i,I} x^I$ for multi-induces I. Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$ be the Veronese embed-

d^{th}

of $\mathbb{P}^n$

ding. Note that each g_i factors through the pullback: $g_i = \nu_d^*(h_i)$, where each h_i is a linear polynomial over $\mathbb{P}^{\binom{n+d}{d}-1}$. Let $Y_i = \mathbb{V}(h_i) \subseteq \mathbb{P}^{\binom{n+d}{d}-1}$ be the corresponding hyperplane. Then $\mathbb{V}(g_i) \cong Y_i \cap \operatorname{im} \nu_d$. Hence

as ν_d is a closed immersion

$$X = \bigcap_{i=1}^{\ell} \mathbb{V}(g_i) \cong \bigcap_{i=1}^{\ell} Y_i \cap \operatorname{im} \nu_d$$

The intersection $\bigcap_{i=1}^{\ell} Y_i$ is a projectivisation of a subspace of $k^{\binom{n+d}{d}}$, and $\operatorname{im} \nu_d$ is a Veronese variety. ✓

READ THE QUESTION!

(b) From the lectures we know that

$$\operatorname{im} \nu_d = \bigcap_{I+J=K+L} \mathbb{V}(x_I x_J - x_K x_L) \checkmark$$

Hence $\operatorname{im} \nu_d$ is an intersection of quadric hypersurfaces. On the other hand, every Y_i is a intersection of hypersurfaces:

Idea: if $\mathbb{P}(V) \subseteq \mathbb{P}(W)$ then (quadric in $\mathbb{P}(W)$) $\cap\, \mathbb{P}(V)$ = (quadric in $\mathbb{P}(V)$) or $\mathbb{P}(V)$

$$Y_i = \mathbb{V}(h_i) = \bigcap_{|J|=d} \mathbb{V}(x_J h_i)$$

Hence the projective variety X is isomorphic to an intersection of some hypersurfaces. □

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Question 5. The ruled surface

The image of the Segre morphism $\sigma_{1,1}\left(\mathbb{P}^1 \times \mathbb{P}^1\right) = \Sigma_{1,1} \subseteq \mathbb{P}^3$ is known as the ruled surface.

(a) What equations define $\Sigma_{1,1}$ as a subvariety of $\mathbb{P}^3$?

(b) What are the images in $\Sigma_{1,1}$ of $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{p\}$? Show that through any point in $\Sigma_{1,1}$ there are two lines lying in $\Sigma_{1,1}$.

(c) Exhibit some disjoint lines in $\Sigma_{1,1}$. Recall that $\mathbb{P}^1 \times \mathbb{P}^1 \cong \Sigma_{1,1}$. Is this isomorphic to $\mathbb{P}^2$? Draw the "real cartoons" of either surface.

Proof. (a) The Segre embedding is given by

$$\sigma_{1,1} \colon \left([x:y],[u:v]\right) \mapsto \left[\begin{pmatrix} xu & xv \\ yu & yv \end{pmatrix}\right]$$

We claim that $\Sigma_{1,1} = \mathbb{V}(z_{00}z_{11} - z_{10}z_{01})$. It is clear that $\Sigma_{1,1} \subseteq \mathbb{V}(z_{00}z_{11} - z_{10}z_{01})$ as $xu \cdot yv - xv \cdot yu = 0$. On the other hand, if $z_{00}z_{11} - z_{10}z_{01} = 0$, then the matrix

$$M = \begin{pmatrix} z_{00} & z_{01} \\ z_{10} & z_{11} \end{pmatrix}$$

has rank 1. Let (x, y) spans the 1-dimensional column space and (u, v) spans the one dimensional row space. Then $M = (x, y)^\top (u, v)$. Hence $\mathbb{V}(z_{00}z_{11} - z_{10}z_{01}) \subseteq \Sigma_{1,1}$. ✓

I'd like the formula for general p please!

(b) By applying a projective transformation we may assume that $p = [1:0]$. Then we have

$$\sigma_{1,1}(\{p\} \times \mathbb{P}^1) = \left\{\left[\begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix}\right] : u, v \in k\right\}, \qquad \sigma_{1,1}(\mathbb{P}^1 \times \{p\}) = \left\{\left[\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}\right] : x, y \in k\right\} \ (\checkmark)$$

These are two projective lines in $\mathbb{P}^3$ intersecting at the point $\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]$.

For $q \in \Sigma_{1,1}$, let $q = \sigma_{1,1}(p_1, p_2)$. Then q lies in the intersection of the projective lines $\sigma_{1,1}(\{p_1\} \times \mathbb{P}^1)$ and $\sigma_{1,1}(\mathbb{P}^1 \times \{p_2\})$ in $\Sigma_{1,1}$. ✓

(c) The following are two disjoint lines in $\Sigma_{1,1}$:

$$\left\{\left[\begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix}\right] : u, v \in k\right\}, \qquad \left\{\left[\begin{pmatrix} 0 & 0 \\ u & v \end{pmatrix}\right] : u, v \in k\right\}$$

Suppose that $\Sigma_{1,1} \cong \mathbb{P}^2$. Then the image of the two disjoint curves under the isomorphism is two disjoint plane curves on $\mathbb{P}^2$. But we know from *B3.4 Algebraic Curves* that any two projective curves on $\mathbb{P}^2$ intersect in at least one point. Contradiction. Hence $\Sigma_{1,1} \ncong \mathbb{P}^2$. □

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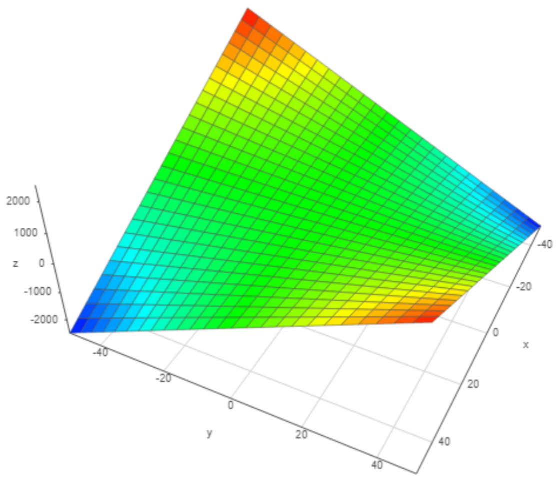

Figure 2: The real slice of an affine patch of the surface $\Sigma_{1,1} \subseteq \mathbb{P}^3$.

Question 6. Rational normal curves

Fix a positive integer $d > 1$.

(a) Let $G(x_0, x_1) = \prod_{i=1}^{d+1}(b_i x_0 - a_i x_1)$ be a homogeneous degree $(d+1)$ polynomial with distinct roots $[a_i : b_i] \in \mathbb{P}^1$. Show that $H_i(x_0, x_1) = G(x_0, x_1)/(b_i x_0 - a_i x_1)$ form a basis for the space of homogeneous polynomials of degree d.

(b) Deduce that the image of the map $\mu_d : \mathbb{P}^1 \to \mathbb{P}^d$ defined by

$$[x_0 : x_1] \longmapsto [H_1(x_0, x_1) : \cdots : H_{d+1}(x_0, x_1)]$$

is projectively equivalent to the image of the Veronese embedding, that is, it is a rational normal curve of degree d.

(c) What is the image of the point $[a_i : b_i]$ under μ_d? If (a_i, b_i) are nonzero for all i, what is the image of $[1:0]$ and $[0:1]$?

(d) Deduce that through any $d+3$ points in general position in $\mathbb{P}^d$, there passes a unique rational normal curve of degree d. (Recall that for $d+3$ points to be in general position means that no subset of $d+1$ of these points lies on a hyperplane in $\mathbb{P}^d$.)

Proof. (a) Let V be the space of homogeneous polynomials of degree d. Then $\dim V = d+1$. So it suffices to check that $\{H_1, \ldots, H_{d+1}\}$ is linearly independent. Suppose that $\sum_{i=1}^{d+1} \lambda_i H_i = 0$. Evaluating at $[a_i : b_i]$ we have $\lambda_i H(a_i, b_i) = 0$. By definition $H_1(a_i, b_i) \neq 0$. Hence $\lambda_i = 0$. This implies that $\{H_1, \ldots, H_{d+1}\}$ is a basis of V.

(b) The Veronese embedding $\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$ is given by

$$[x_0 : x_1] \mapsto \left[x_0^d : x_0^{d-1}x_1 : \cdots : x_0x_1^{d-1} : x_1^d\right]$$

Since $x_0^d, x_0^{d-1}x_1, ..., x_0x_1^{d-1}, x_1^d$ is a basis of V, there exists a change-of-basis endomorphism $T \in \mathrm{GL}(V)$ such that $T(H_i(x_0, x_1)) = x_0^{d-i+1}x_1^{i-1}$. T induces a projective transformation $\widetilde{T} \in \mathrm{PGL}_n(k)$ which maps $\operatorname{im}\mu_d$ to $\operatorname{im}\nu_d$. Hence $\operatorname{im}\mu_d$ is projectively equivalent to the rational normal curve $\operatorname{im}\nu_d$. ✓

(c) We know that $H_j(a_i, b_i) = 0$ for $j \neq i$ and $H_i(a_i, b_i) \neq 0$. Hence

$$\mu_d([a_i, b_i]) = [0 : \cdots : 0 : H_i(a_i, b_i) : 0 : \cdots : 0] = [0 : \cdots : 0 : 1 : 0 \cdots : 0] \quad ✓$$

If $a_i b_i \neq 0$ for all i, then

How come?

$$\mu_d([1:0]) = \left[\frac{1}{b_0} : \cdots : \frac{1}{b_n}\right], \qquad \mu_d([0:1]) = \left[\frac{1}{a_0} : \cdots : \frac{1}{a_n}\right] \quad (✓)$$

(d) By the general position theorem, there exists a projective transformation which put the points in the coordinates

$$\alpha_i = [0 : \cdots : 0 : 1 : 0 \cdots : 0],\ 0 \leqslant i \leqslant d, \qquad \alpha_{n+1} = [1 : \cdots : 1], \qquad \alpha_{n+2} = [c_0 : \cdots : c_n] \quad ✓$$

We claim that the points $[1 : c_0], ..., [1 : c_n]$ are distinct. Suppose that $[1 : c_i] = [1 : c_j]$, then $c_i = c_j$. Then $\alpha_0, ..., \widehat{\alpha_i}, \widehat{\alpha_j}, ..., \alpha_n, \alpha_{n+1}, \alpha_{n+2}$ spans a hyperplane of $\mathbb{P}^n$, which is contradictory to the assumption. Hence the claim is proven. Similarly, we cannot have $c_i = 0$ for some i.

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Now we take $a_i = 1$, $b_i = c_i^{-1}$. Then by the result of part (c), we have $\alpha_0, ..., \alpha_{n+2} \in \operatorname{im}\mu_d$. There exists a rational normal curve passing through $d + 3$ points in general position in $\mathbb{P}^d$. ✓

The uniqueness is much harder to show. The linear algebra technique in the proof of five points determining a conic is not applicable here... Indeed, and the argument involved is a subtle one - see class □

Section C: Optional

Question 7. Projective variety corresponding to a graded ring

If $R = \sum_{d \geqslant 0} R_d$ is a graded ring and $e \geqslant 1$ is an integer, we define

$$R^{(e)} := \sum_{d \geqslant 0} R_{de}$$

We define a grading on $R^{(e)}$ by letting $R_d^{(e)} := R_{de}$.

(a) Find $k[x_0, x_1]^{(2)}$, expressing it in the form $k[z_0, \ldots, z_n]/I$ for some n and I.

(b) Find the homogeneous coordinate rings $S(\mathbb{P}^1)$ and $S(\nu_2(\mathbb{P}^1))$. Comment in the context of part (a).

(c) More generally, show that $S(\nu_e(\mathbb{P}^n)) \cong k[x_0, \ldots, x_n]^{(e)}$, and hence that $k[x_0, \ldots, x_n]^{(e)}$ defines the same projective variety as $k[x_0, \ldots, x_n]$.

(d) Are $k[x_0, \ldots, x_n]^{(e)}$ and $k[x_0, \ldots, x_n]$ isomorphic as graded k-algebras? Are they isomorphic as (ungraded) k-algebras? What does this imply about the affine cones of $\nu_e(\mathbb{P}^n)$ and $\mathbb{P}^n$?

Proof. (a) As a k-algebra, $k[x_0, x_1]^{(2)}$ is generated by x_0^2, x_0x_1, x_1^2. These polynomials satisfy the relation $x_0^2 \cdot x_1^2 - (x_0x_1)^2 = 0$. Hence we have $k[x_0, x_1]^{(2)} \cong k[z_0, z_1, z_2]/\langle z_0z_2 - z_1^2\rangle$. ✓

(b) The homogeneous coordinate ring of $\mathbb{P}^1$ is just the polynomial ring. $S(\mathbb{P}^1) = k[x_0, x_1]$.

✓

Since $\nu_2(\mathbb{P}^1) = \mathbb{V}(z_0 z_2 - z_1^2)$, we have $S(\nu_2(\mathbb{P}^1)) = k[z_0, z_1, z_2] / \langle z_0 z_2 - z_1^2 \rangle$. ✓

From part (a), we note that $S(\nu_2(\mathbb{P}^1)) \cong k[x_0, x_1]^{(2)}$.

NB: this needs justifying, e.g. by invoking (†)

(c) From the lectures we know that

$$\nu_e(\mathbb{P}^n) = \bigcap_{I+J=K+L} \mathbb{V}(z_I z_J - z_K z_L) \quad (\dagger)$$

also need to know that $\langle z_I z_J - z_K z_L \rangle$ is a radical ideal

which implies that

$$S(\nu_e(\mathbb{P}^n)) = \frac{k[x_0, \ldots, x_n]}{\sum_{I+J=K+L} \langle x_I x_J - x_K x_L \rangle} \quad (\checkmark)$$

"generated by the relations"

$k[x_0, \ldots, x_n]^{(e)}$ is generated as a k-algebra by $\{x^I : |I| = e\}$. The relations satisfied by this set are exactly $\{x^I x^J = x^K x^L : I + J = K + L\}$. Hence ✗

e.g. $x^{2I} x^{2J} = x^{2K} x^{2L}$ $(I+J=K+L)$ is another relation, so is $5x^I x^J = 5x^K x^L$ (char $k \neq 5$)

$$S(\nu_e(\mathbb{P}^n)) \cong k[x_0, \ldots, x_n]^{(e)} \quad (\checkmark)$$

Since $\mathbb{P}^n \cong \nu_e(\mathbb{P}^n)$ are projective varieties, we deduce that $k[x_0, \ldots, x_n]$ and $k[x_0, \ldots, x_n]^{(e)}$ are isomorphic homogeneous coordinate rings. They're not; you said this yourself in part d)!

(d) We assume that $e > 1$.

If $k[x_0, \ldots, x_n] \cong k[x_0, \ldots, x_n]^{(e)}$ as graded k-algebras, then $k[x_0, \ldots, x_n]_1 \cong k[x_0, \ldots, x_n]_1^{(e)} = k[x_0, \ldots, x_n]_e$ as Abelian groups. However, we have

$$k[x_0, \ldots, x_n]_1 \cong \mathbb{Z}^{\oplus(n+1)}, \quad k[x_0, \ldots, x_n]_e \cong \mathbb{Z}^{\oplus\binom{n+e}{e}}$$

as Abelian groups. Therefore $k[x_0, \ldots, x_n]$ and $k[x_0, \ldots, x_n]^{(e)}$ are not isomorphic as graded k-algebras. ✓

(*) $k[x_0, \ldots, x_n]$ and $k[x_0, \ldots, x_n]^{(e)}$ are not isomorphic as k-algebras, because $k[x_0, \ldots, x_n]$ is free and $k[x_0, \ldots, x_n]^{(e)}$ is not. ✗ e.g. $\frac{k[x,y,z]}{(xy - z^2)}$ has basis $\{1, x, y, z, x^2, xy, xz, y^2, yz, x^3, \ldots\}$

The result suggests that the affine cone of $\mathbb{P}^n$ and $\nu_e(\mathbb{P}^n)$ are not isomorphic. $\square$

(*): I claim that $k[x_0, \ldots, x_n]^{(e)}$ is not a UFD, unlike $k[x_0, \ldots, x_n]$

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