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Problem Sheet 2
B1.2: Set Theory

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We use the first-order language $\mathcal{L} := \{\in, \subseteq; P, \bigcup, \mathcal{P}; \varnothing, \omega\}$, where $\in$ and $\subseteq$ are binary predicates, P is a binary function, $\bigcup$ and $\mathcal{P}$ are unary functions, and $\varnothing$ and ω are constants.

The equality symbol $\doteq$ is used in $\mathcal{L}$ which indicates that two terms have the same value under any model and assignment. The equality symbol $=$ is used in metalanguage which indicates that two strings are equal.

The ZFC axioms we shall use in this sheet are listed below:

ZF1 *Extensionality:* $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \leftrightarrow x \doteq y)$;

ZF2 *Empty Set:* $\forall x\ \neg x \in \varnothing$;

ZF3 *Pairs:* $\forall x \forall y \forall z (z \in P(x, y) \leftrightarrow (x \doteq z \vee y \doteq z))$;

ZF4 *Unions:* $\forall x \forall y (y \in \bigcup x \leftrightarrow \exists z (y \in z \wedge z \in x))$;

ZF5 *Comprehension Scheme: Let* $\varphi \in \mathrm{Form}(\mathcal{L})$ *and* $z, w_1, ..., w_k \in \mathrm{Free}(\varphi)$. *Then* $\forall x \forall w_1 \cdots \forall w_k \exists y \forall z (z \in y \leftrightarrow (z \in x \wedge \varphi))$;

ZF6 *Power Sets:* $\forall x \forall y (y \in \mathcal{P}(x) \leftrightarrow y \subseteq x)$;

ZF7 *Infinity:* $\exists x (\varnothing \in x \wedge \forall y (y \in x \rightarrow y^+ \in x))$, *where* y^+ *is defined to be* $\bigcup P(y, P(y, y))$;

ZF8 *Replacement Scheme: Let* $\varphi \in \mathrm{Form}(\mathcal{L})$ *and* $x, y, w_1, ..., w_k, A \in \mathrm{Free}(\varphi)$. *Then* $\forall A \forall w_1 \cdots \forall w_k (\forall x (x \in A \rightarrow \exists! y\ \varphi) \rightarrow \exists B \forall y (y \in B \leftrightarrow \exists x (x \in A \rightarrow \varphi)))$, *where* $\exists! y\ \varphi$ *is defined to be* $(\exists y\ \varphi \wedge \forall z \forall y ((\varphi \wedge \varphi[z/y]) \rightarrow y \doteq z))$.

This is ∧!

ZF9 *Foundation:* $\forall x (\neg x \doteq \varnothing \rightarrow \exists y (y \in x \wedge \neg \exists z (z \in y \wedge z \in x)))$;

AC *Choice:* $\forall x (\neg \varnothing \in x \rightarrow \exists f ((f : x \rightarrow \bigcup x) \wedge \forall y (y \in x \rightarrow f(y) \in y)))$, *where* $(f : x \rightarrow \bigcup x)$ *suggests that* f *is a map from* x *to* $\bigcup x$.

The predicate $\subseteq$ *is introduced for convenience. It satisfies* $\forall x \forall y (x \subseteq y \leftrightarrow \forall z (z \in x \rightarrow z \in y))$.

The constant ω *is the smallest inductive set, whose existence and uniqueness follows from the Axioms of Infinity, Comprehension Scheme, and Extensionality.*

Question 1 α

(a) Let R be a relation (i.e. R is a set of ordered pairs). Prove that $\mathrm{Dom}(R)$, which we define to be $\{x : \exists y\ \langle x, y \rangle \in R\}$, is a set.

(b) Let X, Y be sets. Prove there exists a set whose elements are the surjections from X to Y.

(c) Let X be a set. Prove that there is a set consisting precisely of all strict total orders on X.

Proof. (a) Note that the ordered pair $\langle x, y \rangle$ is defined to be $\{\{x\}, \{x, y\}\} = P(P(x, x), P(x, y))$. Since $R \subseteq \mathcal{P}(\mathcal{P}(X \cup Y))$, $\bigcup \bigcup R \subseteq X \cup Y$. Therefore we can use the Axiom Scheme of Comprehension and define the domain to be the set

$$\mathrm{Dom}(R) := \left\{ x \in \bigcup \bigcup R :\ \exists y\ \langle x, y \rangle \in R \right\}$$ ✓

(b) If $f : X \rightarrow Y$ is a map, then $f \subseteq \mathcal{P}(\mathcal{P}(X \cup Y))$. Hence $f \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X \cup Y)))$. We know that it is a first-order property that $f : X \rightarrow Y$ is a surjection:

$$\varphi := (\forall x \forall y (\langle x, y \rangle \in f \rightarrow (x \in X \wedge y \in Y)) \wedge \forall x (x \in X \rightarrow \exists! y\ \langle x, y \rangle \in f) \wedge \forall y (y \in Y \rightarrow \exists x\ \langle x, y \rangle \in f))$$

Therefore we can use the Axiom Scheme of Comprehension and define

$$S := \{f \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X \cup Y))) :\ \varphi\}$$ ✓

S is the set of all surjections $f : X \rightarrow Y$.

(c) Let R be a relation on X, i.e. $R \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$. We know that being a strict total order is a first-order property:

$$\begin{aligned} \varphi_1 :\quad & \forall x \forall y ((\langle x, y \rangle \in R \wedge \neg \langle y, x \rangle \in R \wedge \neg x \doteq y) \\ & \vee (\neg \langle x, y \rangle \in R \wedge \langle y, x \rangle \in R \wedge \neg x \doteq y) \\ & \vee (\neg \langle x, y \rangle \in R \wedge \neg \langle y, x \rangle \in R \wedge x \doteq y)) && \text{[Trichotomy]} \\ \varphi_2 :\quad & \forall x \forall y \forall z (\langle x, y \rangle \in R \rightarrow (\langle y, z \rangle \in R \rightarrow \langle x, z \rangle \in R)) && \text{[Transitivity]} \end{aligned}$$

Therefore we can use the Axiom Scheme of Comprehension and define

$$S := \{R \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X))) : \varphi_1 \wedge \varphi_2\}$$

✓

S is the set of all strict total orders on X. □

Question 2 α

Prove, using induction and the fact that each $n \in \omega$ is transitive, that $n \in n$ is false for every $n \in \omega$.

[*Do not use the Axiom of Foundation.*]

Proof. Base case: Since $\varnothing \notin \varnothing$, we have $0 \notin 0$.

Induction case: Suppose that $n \notin n$. By definition, $n^+ := n \cup \{n\}$. Assume that $n^+ \in n^+$. Then $n \cup \{n\} \in n \cup \{n\}$. So $n \cup \{n\} \in n$ or $n \cup \{n\} \doteq n$.

If $n \cup \{n\} \doteq n$, then $\{n\} \subseteq n$. Then $n \in n$, contradicting the induction hypothesis.

If $n \cup \{n\} \in n$, since n is transitive, then $n \cup \{n\} \subseteq n$. Therefore $\{n\} \subseteq n$. Then $n \in n$, contradicting the induction hypothesis.

Hence $n^+ \notin n^+$. By Theorem 4.6, $n \notin n$ for all $n \in \omega$. ✓ □

Question 3 α-

Prove that the function sending n to $n!$ exists (as a set).

[*Hint: Use Recursion in the form of Theorem 5.1 with* $X = \omega \times \omega$.]

Proof. Let $g : \omega \times \omega \to \omega \times \omega$ given by $\langle m, n\rangle \mapsto \langle m \cdot n^+, n^+\rangle$. The map g is well-defined. Now fix $\langle 1, 0\rangle \in \omega$. By Theorem 5.1 there exists a map $f : \omega \to \omega \times \omega$ such that $f(0) \doteq \langle 1, 0\rangle$ and $f(n^+) \doteq g(f(n))$.

By the Axiom Scheme of Comprehension, there exists a set

$$\tilde{f} := \{\langle x, y\rangle \in \omega \times \omega : \forall z \ \langle x, \langle y, z\rangle\rangle \in f\}$$

∃z instead. You only have ⟨x, ⟨y, x⟩⟩ ∈ f, not ⟨x, ⟨y, z⟩⟩ ∈ f for arbitrary z

We claim that $\tilde{f}$ is a map from ω to ω such that every $n \in \omega$ is sent to $n! \in \omega$. □

Question 4 α

Prove that multiplication on ω is commutative by proving the following statements for $n, m \in \omega$ by induction. You may use the other arithmetic properties established in lectures; or prove that $m * n = n \cdot m$ satisfies the same recursion as $m \cdot n$.

(i) $0 \cdot n = 0$

(ii) $m^+ \cdot n = m \cdot n + n$

(iii) $m \cdot n = n \cdot m$

Proof. We use the following properties of the multiplication $\cdot : \omega \times \omega \to \omega$.

1. $n \cdot 0 \doteq 0$ for all $n \in \omega$;
2. $n \cdot m^+ \doteq n \cdot m + n$ for all $n, m \in \omega$;
3. Any map $\omega \times \omega \to \omega$ with the above properties is unique.

To prove (i) - (iii) you do not actually need this. You need this only if you view in the alternative way (proving that ⟨m, n⟩ → m · n and ⟨m, n⟩ → n · m are the same map)

In addition, we assume all arithmetic of the addition $+ : \omega \times \omega \to \omega$.

(i) We use induction on n.

Base case: $0 \cdot 0 \doteq 0$.

Induction case: Suppose that $0 \cdot n \doteq 0$. Then $0 \cdot n^+ \doteq 0 \cdot n + 0 \doteq 0 + 0 \doteq 0$. ✓

(ii) We use induction on n.

Base case: $m^+ \cdot 0 \doteq 0 \doteq 0 + 0 \doteq m \cdot 0 + 0$.

Induction case: Suppose that $m^+ \cdot n \doteq m \cdot n + n$. Then

$$m^+ \cdot n^+ \doteq m^+ \cdot n + m^+ \doteq (m \cdot n + n) + (m + 1) \doteq (m \cdot n + m) + (n + 1) \doteq m \cdot n^+ + n^+$$ ✓

(iii) We use induction on m.

Base case: $0 \cdot n \doteq 0$.

Induction case: Suppose that $m \cdot n \doteq n \cdot m$. Then

$$m^+ \cdot n \doteq m \cdot n + n \doteq n \cdot m + n \doteq n \cdot m^+$$ ✓ □

Question 5 α-

Write $1 = 0^+, 2 = 1^+$. Define $n \in \omega$ to be *even* if it is of the form $2 \cdot k$ for some $k \in \omega$ and *odd* if it is of the form $2 \cdot h + 1$ for some $h \in \omega$. Prove that

(i) every element of ω is either even or odd;

(ii) no element of ω is both even and odd.

Proof. (i) We use induction on n to prove that every $n \in \omega$ is either even or odd.

Base case: $0 \in \omega$ is even. This is because $0 \doteq 2 \cdot 0$.

Induction case: Suppose that $n \in \omega$ is either even or odd.

If n is even, then there exists $k \in \omega$ such that $n \doteq 2 \cdot k$. Then $n^+ \doteq (n+0)^+ \doteq n + 0^+ \doteq n + 1 \doteq 2 \cdot k + 1$ is odd.

If n is odd, then there exists $k \in \omega$ such that $n \doteq 2 \cdot k + 1$. Then

$$n^+ \doteq n + 1 \doteq (2 \cdot k + 1) + 1 \doteq 2 \cdot k + (1 + 1) \doteq 2 \cdot k + (1 + 0^+) \doteq 2 \cdot k + (1 + 0)^+ \doteq 2 \cdot k + 1^+ \doteq 2 \cdot k + 2 \doteq 2 \cdot k^+$$

is even.

Hence n^+ is either even or odd. ✓

(ii) Suppose for contradiction that $n \in \omega$ is both even and odd. There exists $k, h \in \omega$ such that $n = 2 \cdot k = 2 \cdot h + 1$.

First we use induction on m to prove that $\varphi_m := (k \in m \to \exists \ell(\ell \in \omega \wedge k + \ell \doteq m))$.

Base case: The formula φ_0 holds vacuously. No need for this. m+ = m ∪ {m}, so k ∈ m+ implies either k = m or k ∈ m.

Induction case: Suppose that φ_m holds. If $m \in k$, then $k \notin m^+$ because $\in$ is a strict total order on ω. φ_{m^+} holds vacuously. If $m \doteq k$, then $m^+ \doteq k^+$. We have $k + 1 \doteq k^+ \doteq m^+$, so φ_{m^+} holds. If $k \in m$, then by induction hypothesis there exists $\ell \in \omega$ such that $k + \ell \doteq m$. Hence $k + (\ell + 1) \doteq (k + \ell) + 1 \doteq m + 1 \doteq m^+$. So φ_{m^+} holds. ✓

Next, we consider the three cases.

If $k \doteq h$, then $2 \cdot k \doteq 2 \cdot h + 1$ implies that $0 \doteq 1$, which is impossible.
Too quick here! To cancel, you need to mention/prove that + is injective on the second operand

If $k \in h$, then there exists $\ell \in \omega$ such that $k + \ell = h$. Therefore

$$2 \cdot k \doteq 2 \cdot h + 1 \implies 2 \cdot k \doteq 2 \cdot (k + \ell) + 1 \implies 2 \cdot k \doteq 2 \cdot k + 2 \cdot \ell + 1 \implies 0 \doteq 2 \cdot \ell + 1 \implies 0 \doteq (2 \cdot \ell)^+$$

Distribution — this need another induction if you do not state it priorly

But $0 = \varnothing$ is not the successor of any set. Contradication.

If $h \in k$, then there exists $\ell \in \omega$ such that $h + \ell = k$. Therefore

$$2 \cdot k \doteq 2 \cdot h + 1 \implies 2 \cdot \ell \doteq 1$$

We shall prove that this is impossible by induction on ℓ. Consider the formula $\psi_\ell := (\ell \doteq 0 \vee 1 \in 2 \cdot \ell)$.

A much simpler way for this part, directly use induction on n:
- For n = 0: 2 · k + 1 = (2 · k)+ ≠ 0, so 0 cannot be odd
- Assume n, for n+: suppose n+ = 2 · k = 2 · h + 1, then k ≠ 0, so k = m+, and n = 2 · m + 1 = 2 · h, contradicting inductive hypothesis

The base case ψ_0 is trivial. Suppose that ψ_ℓ holds. If $\ell \doteq 0$, then $1 \in 2 \doteq 2 \cdot 1 \doteq 2 \cdot \ell^+$. So ψ_{ℓ^+} holds. If $0 \in \ell$, then by induction hypothesis, $1 \in 2 \cdot \ell$. But $2 \cdot \ell \in 2 \cdot \ell + 2 \doteq 2 \cdot \ell^+$. So $1 \in 2 \cdot \ell^+$, and ψ_{ℓ^+} holds. ✓ □

Question 6 α-

A *Peano system* is a triple (A, s, a_0) in which A is a set, $a_0 \in A$, and $s : A \to A$ is a function which is (a) one-to-one, (b) does not include a_0 in its range, and (c) satisfies the Principle of Induction: that is, if $S \subseteq A, a_0 \in S$ and $\forall a(a \in S \to s(a) \in S)$, then $S = A$.

(i) Prove that $(\omega, x \mapsto x^+, 0)$ is a Peano system.

(ii) Suppose (A, s, a_0) is a Peano system. Prove that there exists an isomorphism from $(\omega, {}^+, 0)$ to (A, s, a_0), that is, there is a bijection $f : \omega \to A$ such that $f(0) = a_0$ and, for all $n \in \omega$, $f(n^+) = s(f(n))$.

[*Hence, up to isomorphism,* $(\omega, {}^+, 0)$ *is the unique Peano system.*]

[*Hint*: *Define f by recursion and verify the required properties.*]

Proof. (i) Let $s : \omega \to \omega$ defined by $x \mapsto x^+$.

First, by Theorem 4.11, s is injective.

Second, it is clear that $n \notin 0 = \varnothing$ for all $n \in \omega$. Hence $0 \notin \operatorname{im}(s)$.

Third, suppose that $S \subseteq \omega$, $0 = \varnothing \in S$, and $\forall a(a \in S \to a^+ \in S)$. The by Axiom of Infinity S is an inductive set. Hence $\omega \subseteq S$. But $S \subseteq \omega$. So $S \doteq \omega$. ✓

You do not need Axiom of Infinity here

(ii) By Theorem 5.1 there exists a map $f : \omega \to A$ such that $f(0) \doteq a_0$ and $f(n^+) \doteq s(f(n))$. We shall prove that f is a bijection.

f is surjective: Consider $S := \operatorname{im}(f) := \{y \in A : \exists n(n \in \omega \wedge y \doteq f(n))\}$. Since $a_0 = f(0)$, $a_0 \in S$. For $a \in S$, there exists $n \in \omega$ such that $a = f(n)$. Then $s(a) = s(f(n)) = f(n^+) \in S$. Hence $\forall a(a \in S \to s(a) \in S)$. As (A, s, a_0) is a Peano system, we conclude that $S \doteq A$. Hence f is surjective. ✓

f is injective: Suppose that it is not. There exists $n, m \in \omega$ such that $n \neq m$ and $f(n) \doteq f(m)$. Without loss of generality suppose that $n \in m$. Then there exists $\ell \in \omega$ such that $n + \ell \doteq m$. Then $f(n + \ell) \doteq f(n)$.

(a) We prove by induction on k that $f(n + k) \doteq f(m + k)$ for all $k \in \omega$:

Base case: $f(n + 0) \doteq f(n) \doteq f(m) \doteq f(m + 0)$.

Induction case: Suppose that $f(n + k) \doteq f(m + k)$. Then $f(n + k^+) \doteq f((n + k)^+) \doteq s(f(n + k)) \doteq s(f(m + k)) \doteq f((m + k)^+) \doteq f(m + k^+)$. ✓

(b) We prove by induction on k that $f(n + k \cdot \ell) \doteq f(n)$ for all $k \in \omega$:

This is not by inductive hypothesis here, but rather f(m + k · l) = f(n + k · l) from (a)! So you should write instead:
f(n + k · l + l) = f(m + k · l) = f(n + k · l) = f(n)

Base case: $f(n + 0 \cdot \ell) \doteq f(n + 0) \doteq f(n)$.

Induction case: Suppose that $f(n + k \cdot \ell) \doteq f(n)$. Then $f(n + k^+ \cdot \ell) \doteq f(n + k \cdot \ell + \ell) \doteq f(n + \ell) \doteq f(n)$.

(c) We prove by induction on a that $\varphi_a := ((a \in \omega \wedge n \in a) \to \exists k(k \in \omega \wedge n + k \cdot \ell \leqslant a \wedge a \in n + k^+ \cdot \ell))$.

Base case: φ_0 is true vacuously.

Again, no need for this. a+ = a ∪ {a}, so n ∈ a+ implies either n = a or n ∈ a.

Induction case: Suppose that φ_a is true. If $a \in n$, then $n \notin a^+$, so φ_{a^+} holds vacuously. If $a \doteq n$, then $n + 0 \cdot \ell \leqslant n \doteq a$ and $a \doteq n \in n + 1 \cdot \ell$. If $n \in a$, by induction hypothesis there exists $k \in \omega$ such that $k \in \omega \wedge n + k \cdot \ell \leqslant a$ and $a \in n + k^+ \cdot \ell$. Let $b \in \omega$ such that $a + b \doteq n + k^+ \cdot \ell$. If $b \doteq 1$, then $a^+ \doteq n + k^+ \cdot \ell$ and $a^+ \in n + k^{++} \cdot \ell$; if $1 \in b$, then $a^+ \in n + k^+ \cdot \ell$ and $n + k \cdot \ell \leqslant a^+$. ✓

(d) Now, for $a \in \omega$ such that $n \in a$, there exists $k \in \omega$ such that $n + k \cdot \ell \leqslant a < n + k^+ \cdot \ell$. Let $b \in \omega$ such that $a \doteq n + k \cdot \ell + b$. In particular, since $a \in n + k^+ \cdot \ell$, we have $b \in \ell$. Then $f(a) \doteq f(n + k \cdot \ell + b) \doteq f(n + b)$. We deduce that $\operatorname{im}(f)$ has at most $m - 1$ distinct elements: $f(0), ..., f(m-1) = f(n + \ell - 1)$. But since f is surjective, A is a set with finitely many distinct elements.

You have not shown this! Same proof as (a) though, but you need to mention it

(e) Since A is finite, by Theorem 9.2, $s : A \to A$ is injective implies that s is surjective, which contradicts that $a_0 \notin \operatorname{im}(s)$.

We conclude that f is bijective. ✓ □

Again, a much simpler way for this part, directly use induction on n to prove that for all m > n, f(m) ≠ f(n):
- For n = 0: for m > 0, m = k+, so f(m) = s(f(k)) ≠ a0 = f(0)
- Assume n, for n+: for m > n+, m = k+ for some k > n. If s(f(k)) = f(m) = f(n+) = s(f(n)), then f(k) = f(n), contradicting inductive hypothesis

Question 7 α

Let $X = X_0$ be a set. By the Axiom of Unions, the sets $X_1 = \bigcup X, X_2 = \bigcup X_1, \ldots$ are sets. The *transitive closure* of X is defined to be $T(X) = \bigcup_{n=0}^{\infty} X_n = \bigcup \{X_0, X_1, \ldots\}$. Prove that

(i) $T(X)$ is a set

(ii) $T(X)$ is transitive

(iii) $X \subseteq T(X)$

(iv) If $X \subseteq Y$ and Y is transitive then $T(X) \subseteq Y$

(v) If X is transitive then $T(X) = X$.

Proof. (i) Let $\varphi(x, y) := y \doteq \bigcup x$. Then $\forall x \exists! y \varphi(x, y)$. By Theorem 7.2 (which is based on the Axiom Scheme of Replacement), there exists a set Y and a function $f : \omega \to Y$ such that $f(0) \doteq X_0$, $f(1) \doteq \bigcup X_0 \doteq X_1$,... $f(n) \doteq X_n$ for all $n \in \omega$. Then by Axiom Scheme of Comprehension,

$$Z := \{y \in Y : \exists n(n \in \omega \wedge f(n) \doteq y)\}$$

is a set with elements $X_0, X_1, X_2, \ldots$

We can define $T(X)$ to be $\bigcup Z$. In particular, we have $\forall x(x \in T(X) \leftrightarrow \exists n(n \in \omega \wedge x \in X_n))$.

(ii) For $x \in T(X)$, there exists $X_i \in Z$ such that $x \in X_i$. Then $x \subseteq \bigcup X_i = X_{i+1}$. But $X_{i+1} \in Z$ implies that $X_{i+1} \subseteq T(X)$. Hence $x \subseteq T(X)$. Hence $T(X)$ is transitive.

(iii) $X_0 \in Z$ implies that $X_0 \subseteq T(X)$.

(iv) We use induction on n to prove that $X_n \subseteq Y$ for all $n \in \omega$. Base case $X = X_0 \subseteq Y$. Induction case: Suppose that $X_n \subseteq Y$. For $x \in X_{n^+} = \bigcup X_n$, there exists $y \in X_n$ such that $x \in y$. By induction hypothesis $y \in Y$. Since Y is transitive, $x \in y \in Y$ implies that $x \in Y$. Hence $X_{n^+} \subseteq Y$.

We deduce that $T(X) = \bigcup_{n \in \omega} X_n \subseteq Y$.

(v) Let $Y = X$ in (iv) we obtain that $T(X) \subseteq X$. But by (iii) we have $X \subseteq T(X)$. Hence $T(X) \doteq X$. □

Question 8 α

A set X is called *hereditarily finite* if its transitive closure $T(X)$ is a finite set.

(i) Prove that the following sets are hereditarily finite

$$\varnothing, \quad \{\varnothing\}, \quad \{\varnothing, \{\varnothing\}\}, \quad \{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$$

(ii) Prove that a subset of a hereditarily finite set is hereditarily finite, and an element of a hereditarily finite set is hereditarily finite.

[*You may assume: a subset of a finite set is finite*]

(iii) Let $\mathbf{H}$ be the class of hereditarily finite sets. It turns out that $\mathbf{H}$ is in fact a set. Prove that the Empty Set Axiom, the Axioms of Extensionality, Pairs, Unions and the Comprehension Scheme are all true in $\mathbf{H}$.

[*For example, the Axiom of Pairs is true in* $\mathbf{H}$ *provided that, if* a, b *are hereditarily finite sets, there is a hereditarily finite set* c *whose only hereditarily finite elements are a and b. This will be true if indeed* $\{a, b\}$ *is hereditarily finite.*]

(iv) Is $\omega \in \mathbf{H}$?

(v) Show that the Axiom of Infinity is not a consequence (in first order predicate logic) of the Axioms of Empty Set, Extensionality, Pairs, Union and Comprehension Scheme.

(vi) A set is called *hereditarily countable* if $T(X)$ is a countable set (i.e. is finite or is in bijection with ω). Let $\mathbf{K}$ be the class of hereditarily countable sets. In fact $\mathbf{K}$ is a set. Now $\omega \in \mathbf{K}$. Which of the axioms Extensionality, Empty Set, Pairs, Unions, Comprehension Scheme, Infinity, Power Set hold in $\mathbf{K}$?

[*You may use that a countable union of countable sets is countable, though this does not follow from the axioms so far given.*]

(vii) Is it possible to prove the Power Set Axiom from the above axioms (now including Axiom of Infinity)?

Proof. (i) In Sheet 1 Question 1.(i) Line 13, we have proven that $\bigcup \varnothing \doteq \varnothing$. That is, $\bigcup 0 \doteq 0$. We know that ω is transitive. Then $\bigcup n^+ \doteq n$ for all $n \in \omega$. This does not help. You used instead that each n ∈ ω is transitive

Let $X_0 := \varnothing = 0$ and $X_{n^+} := \bigcup X_n$ for each $n \in \omega$. Then $X_n \doteq 0$ for all $n \in \omega$. In particular $T(X) = \bigcup_{n\in\omega} X_n \doteq 0$. Since 0 is finite, we conclude that 0 is hereditarily finite.

Let $X_0 := \{\varnothing\} = 1.$ and $X_{n^+} := \bigcup X_n$ for each $n \in \omega$. Then $X_n \doteq 0$ for all $n \geqslant 1$. In particular $T(X) = \bigcup_{n\in\omega} X_n \doteq 1$. Since 1 is finite, we conclude that 1 is hereditarily finite.

Let $X_0 := \{\varnothing, \{\varnothing\}\} = 2.$ and $X_{n^+} := \bigcup X_n$ for each $n \in \omega$. Then $X_1 \doteq 1$ and $X_n \doteq 0$ for all $n \geqslant 2$. In particular $T(X) = \bigcup_{n\in\omega} X_n \doteq 2$. Since 2 is finite, we conclude that 2 is hereditarily finite.

Let $X_0 := \{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} = \{1,2\}.$ and $X_{n^+} := \bigcup X_n$ for each $n \in \omega$. Then $X_1 = \bigcup\{1,2\} = 2$, $X_2 \doteq \bigcup 2 \doteq 1$ and $X_n \doteq 0$ for all $n \geqslant 3$. In particular $T(X) = \bigcup_{n\in\omega} X_n \doteq \{0,1,2\} = 3$. Since 3 is finite, we conclude that $\{1,2\}$ is hereditarily finite. ✓

(ii) We state a lemma:

$$\forall x \forall y (x \subseteq y \to \bigcup x \subseteq \bigcup y)$$

The proof is trivial.

Let X be hereditarily finite and $Y \subseteq X$. Let $X_0 := X$ and $X_{n^+} := \bigcup X_n$ for all $n \in \omega$. Let $Y_0 := Y$ and $Y_{n^+} := \bigcup Y_n$ for all $n \in \omega$. Inductively we have $Y_n \subseteq X_n$ for all $n \in \omega$. Hence $T(Y) \doteq \bigcup_{n\in\omega} Y_n \subseteq \bigcup_{ni\omega} X_n \doteq T(X)$. Since $T(X)$ is finite, $T(Y)$ is finite. Hence $Y \in \mathbf{H}$.

For $x \in X$, $x \subseteq X_1 = \bigcup X$. Similar to above we have $T(x) \subseteq T(\bigcup X)$. But $T(\bigcup X) \cup X \doteq T(X)$. Hence $T(\bigcup X)$ is finite and $T(x)$ is finite. $x \in \mathbf{H}$. ✓

(iii) $\vDash_{\mathbf{H}}$ ZF1: Trivial.

Note: when a formula φ is true in a model H without needing premises, we can indeed write ⊨_H φ, but more often we actually directly write H ⊨ φ. :)

$\vDash_{\mathbf{H}}$ ZF2: In part (i) We have proven that $\varnothing \in \mathbf{H}$.

$\vDash_{\mathbf{H}}$ ZF3: Suppose that $x, y \in \mathbf{H}$. It is clear that $T(x) \cup T(y) \doteq T(x \cup y)$. Since $x \cup y = \bigcup\{x,y\}$, $T(\{x,y\}) \doteq T(x \cup y) \cup \{x,y\}$. As $T(x)$ and $T(y)$ are finite, we deduce that $T(\{x,y\})$ is finite. Hence $\{x,y\} \in \mathbf{H}$.

$\vDash_{\mathbf{H}}$ ZF4: Suppose that $x \in \mathbf{H}$. Then $T(x)$ is finite. Note that $T(x) \doteq T(\bigcup x) \cup x$. So $T(\bigcup x) \subseteq T(x)$ is finite. Hence $\bigcup x \in \mathbf{H}$.

$\vDash_{\mathbf{H}}$ ZF5: Suppose that $x \in \mathbf{H}$, $\varphi \in \text{Form}(\mathcal{L})$ and $z, w_1, ..., w_k \in \text{Free}(\varphi)$. Let $y := \{z \in x : \varphi\}$. Then $y \subseteq x$. By part (ii) we have $y \in \mathbf{H}$. ✓

(iv) $\omega \subseteq T(\omega)$. If $\omega \in \mathbf{H}$, then $T(\omega)$ is finite. But ω is not finite, contradication. ✓

(v) Suppose that $\{\text{ZF1}, \text{ZF2}, \text{ZF3}, \text{ZF4}, \text{ZF5}\} \vdash \text{ZF7}$. Then by Gödel's Completeness Theorem $\{\text{ZF1}, \text{ZF2}, \text{ZF3}, \text{ZF4}, \text{ZF5}\} \vDash \text{ZF7}$. In part (iii) we have proven that $\vDash_{\mathbf{H}} \{\text{ZF1}, \text{ZF2}, \text{ZF3}, \text{ZF4}, \text{ZF5}\}$. But since $\omega \notin \mathbf{H}$, we have $\nvDash_{\mathbf{H}}$ ZF7. Hence $\{\text{ZF1}, \text{ZF2}, \text{ZF3}, \text{ZF4}, \text{ZF5}\} \nvDash \text{ZF7}$. Contradication. ✓

(vi) An immediate corollary of Theorem 9.10 is that a subset of a countable set is countable. We assume the Axiom of Choice, which implies that a countable union of countable sets is countable.

$\vDash_{\mathbf{K}}$ ZF1: Trivial.

$\vDash_{\mathbf{K}}$ ZF2: In part (i) We have proven that $\varnothing \in \mathbf{H}$. Trivially $\mathbf{H} \subseteq \mathbf{K}$. Then $\varnothing \in \mathbf{K}$.

$\vDash_{\mathbf{K}}$ ZF3: Suppose that $x, y \in \mathbf{K}$. $T(\{x,y\}) \doteq T(x \cup y) \cup \{x,y\}$. As $T(x)$ and $T(y)$ are countable, we deduce that $T(\{x,y\})$ is countable. Hence $\{x,y\} \in \mathbf{K}$.

$\vDash_{\mathbf{K}}$ ZF4: Suppose that $x \in \mathbf{K}$. Then $T(x)$ is countable. Note that $T(x) \doteq T(\bigcup x) \cup x$. So $T(\bigcup x) \subseteq T(x)$ is countable. Hence $\bigcup x \in \mathbf{K}$.

$\vDash_{\mathbf{K}}$ ZF5: Suppose that $x \in \mathbf{K}$, $\varphi \in \text{Form}(\mathcal{L})$ and $z, w_1, ..., w_k \in \text{Free}(\varphi)$. Let $y := \{z \in x : \varphi\}$. Then $y \subseteq x$. As in part (ii) we have $T(y) \subseteq T(x)$. Since $T(x)$ is countable, $T(y)$ is countable. Hence we have $y \in \mathbf{K}$.

$\vDash_{\mathbf{K}}$ ZF7: ω is countable and transitive. Then $T(\omega) \doteq \omega$ is countable. Hence $\omega \in \mathbf{K}$.

$\not\models_{\mathbf{K}}$ ZF6: By Cantor's Theorem, $\omega \prec \mathcal{P}(\omega)$. In particular $\mathcal{P}(\omega)$ is uncountable. As $\mathcal{P}(\omega) \subseteq T(\mathcal{P}(\omega))$, $T(\mathcal{P}(\omega))$ is uncountable. Then $\mathcal{P}(\omega) \notin \mathbf{K}$ whereas $\omega \in \mathbf{K}$. ✓

(vii) Suppose that {ZF1, ZF2, ZF3, ZF4, ZF5, ZF7} $\vdash$ ZF6. Then by Gödel's Completeness Theorem {ZF1, ZF2, ZF3, ZF4, ZF5, ZF7} $\models$ ZF6. In part (vi) we have proven that $\models_{\mathbf{K}}$ {ZF1, ZF2, ZF3, ZF4, ZF5, ZF7} and $\not\models_{\mathbf{K}}$ ZF6. Hence {ZF1, ZF2, ZF3, ZF4, ZF5, ZF7} $\not\models$ ZF6. Contradication. ✓ □