Topology & Groups

Michaelmas 2016

Question Sheet 1

1. For each of the following groups G and generating sets S, draw the resulting Cayley graph:

   (i) $G = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, $S = \{(1,0),(0,1)\}$;

   (ii) $G =$ the dihedral group of order 8, viewed as the symmetries of the square, and $S = \{\sigma, \tau\}$, where σ is a rotation of order 4, and τ is a reflection through an axis joining opposite sides;

   (iii) $G =$ the free group on two generators a and b, and $S = \{a, b\}$. [You will need to be familiar with free groups from Part A Group Theory for this question. If you did not do that course, then skip this part of the question.]

2. Let Γ be the Cayley graph of a group G with respect to a generating set S.

   (i) Show that the following is a metric on G: $d(g_1, g_2) =$ the shortest number of edges in a path in Γ joining the vertex labelled g_1 to the vertex labelled g_2.

   (ii) Show that $d(g_1, g_2)$ equals the smallest non-negative integer n such that

   $$g_2 = g_1 s_1^{\epsilon_1} s_2^{\epsilon_2} \dots s_n^{\epsilon_n}$$

   where each $s_i \in S$ and $\epsilon_i \in \{-1, 1\}$.

   (iii) Let $\mathrm{Isom}(G)$ be the group of isometries of G with this metric. Prove that G can be realised as a subgroup of $\mathrm{Isom}(G)$.

   (iv) Find an example where $G \subsetneqq \mathrm{Isom}(G)$.

3. Recall that the surface S_g with g handles can be constructed from a $4g$-sided polygon by identifying its sides in pairs. Show that S_g can be given the structure of a cell complex, with a single 0-cell, $2g$ 1-cells and a single 2-cell.

4. Give a cell structure for the 3-torus $S^1 \times S^1 \times S^1$. Try to use as few cells as possible. [It is possible to use a single 0-cell, three 1-cells, three 2-cells and one 3-cell.]

Topology & Groups 1

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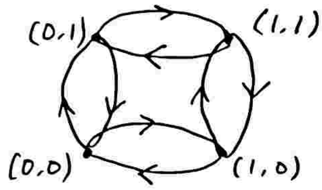
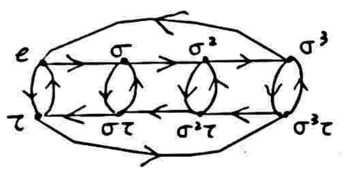
1. (i) $G := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{(0,0), (1,0), (0,1), (1,1)\}$

The Cayley graph with respect to $\{(1,0), (0,1)\}$ is given by :

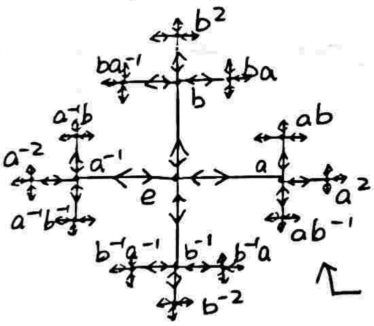


(ii) $G := \{e, \sigma, \sigma^2, \sigma^3, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\}$. We know that $\sigma^4 = \tau^2 = e$

The Cayley graph with respect to $\{\sigma, \tau\}$ is given by :



(iii) G is an infinite group. The Cayley Graph around e looks like :



(ignore all arrows of the form "$\leftarrow$" and "$\downarrow$")

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2. (i) We need to verify the definition for a metric.
The positivity and symmetry are trivial.
For the triangular inequality, we should have $d(g_1, g_2) \leq d(g_1, g_3) + d(g_2, g_3)$.
This is obvious because the concatenation of a path joining g_1 and g_3, and a path joining g_3 and g_2, is a path joining g_1 and g_2. a whose length is ≤ RHS

(ii) First, if there exists $s_1, \cdots, s_n \in S$ and $\varepsilon_1, \cdots, \varepsilon_n \in \{-1, 1\}$ such that $g_2 = g_1 s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}$. Then there exists a concatenation of paths $g_1 - g_1 s_1^{\varepsilon_1} - g_1 s_1^{\varepsilon_1} s_2^{\varepsilon_2} - \cdots - g_1 s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}$ (the arrow direction depends on the sign of ε_i). Hence $d(g_1, g_2) \leq n$.
If $d(g_1, g_2) = m < n$, then there exists a path $g_1 - g_1 s_1^{\varepsilon_1} - \cdots - g_1 s_1^{\varepsilon_1} \cdots s_m^{\varepsilon_m} = g_2$, which contradicts the minimality of n. Hence $d(g_1, g_2) = n$.

(iii) Consider the group action $\sigma : G \times G \to G$ defined by left multiplication.
$\forall g \in G$, σ induces a group automorphism $\sigma_g : h \mapsto gh$.
This is also an isometry of G:
$\forall h_1, h_2 \in G : d(h_1, h_2) = n \Leftrightarrow \exists s_1, \cdots, s_n \in S \ \exists \varepsilon_1, \cdots, \varepsilon_n \in \{1, -1\} \ h_2 = h_1 s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}$

~~why is this subgroup isomorphic to G?~~

$\Leftrightarrow \exists s_1, \cdots, s_n \in S \ \exists \varepsilon_1, \cdots, \varepsilon_n \in \{1, -1\} \ gh_2 = gh_1 s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}$
with n be the smallest such number
$\Leftrightarrow d(gh_1, gh_2) = n$.

* so ϕ is injective

Hence $d(gh_1, gh_2) = d(h_1, h_2)$. σ_g is an isometry.
Moreover, it is obvious that $\sigma_g \circ \sigma_h = \sigma_{gh}$, $\sigma_{g^{-1}} = (\sigma_g)^{-1}$ and that $\sigma_e = id \in Isom(G)$*. Hence $G \leq Isom(G)$ if we identify g with σ_g.

(iv) Consider the Cayley graph given in Q 1.(i).
The following mapping is an isometry:
$(0,0) \mapsto (0,0)$; $(1,0) \mapsto (0,1)$; $(0,1) \mapsto (1,0)$; $(1,1) \mapsto (1,1)$.
But this is not a mapping by left multiplying a $g \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
In this case we have $G \subsetneqq Isom(G)$. ✓ Great

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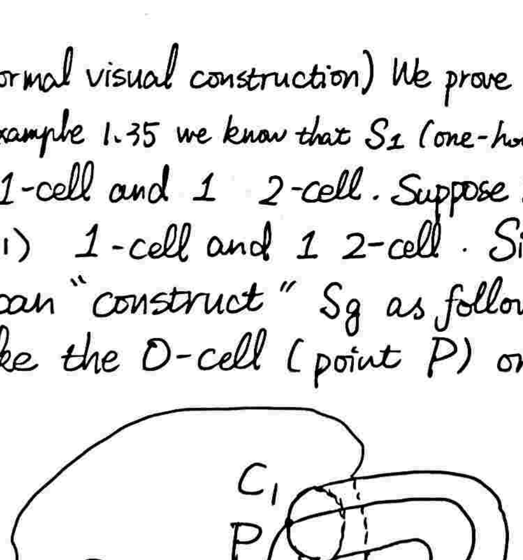
3. (Informal visual construction) We prove by induction:
In Example 1.35 we know that S_1 (one-holed torus) can be constructed by 1 0-cell, 2 1-cell and 1 2-cell. Suppose S_{g-1} can be constructed by 1 0-cell, $2(g-1)$ 1-cell and 1 2-cell. Since S_g is obtained by adding a handle, we can "construct" S_g as follows:
Take the 0-cell (point P) on S_{g-1}: We attach 2 1-cells, whose images are C_1 and C_2, and a 2-cell. But the union of the two 2-cell is in fact simply-connected and is homeomorphic to D^2.
Hence we only need one 2-cell to construct S_g. ✓



(We may also obtain this by identifying pairs of edges of a $4g$-gon.)
→ this approach lets you be precise about orientation

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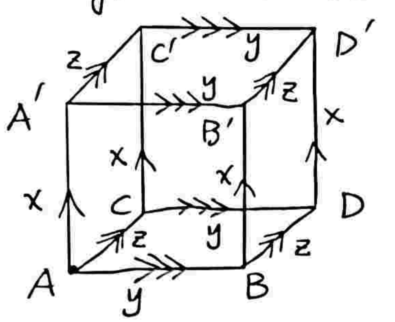
4. We can construct a 3-torus by gluing a cube.

Explicitly, we have the side identification given as follows:



0-cell : Point A

1-cells : The circles represented by the words x, y, z

2-cells : The surfaces with boundary words $xyx^{-1}y^{-1}$, $yzy^{-1}z^{-1}$, $zxz^{-1}x^{-1}$

3-cells : The whole cube

✓

Here is an explicit construction without using quotient topology:

A natural embedding $T^3 := S^1 \times S^1 \times S^1 \hookrightarrow \mathbb{R}^6$ is given by:

$$T^3 = \{(\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2, \cos\theta_3, \sin\theta_3) : \theta_1, \theta_2, \theta_3 \in [0, 2\pi]\}$$

We start from a 0-cell $P = (1, 0, 1, 0, 1, 0)$.

We attach 3 1-cells by the following mappings:

$f_1 : (0, 2\pi) \to \mathbb{R}^6$, $\theta_1 \mapsto (\cos\theta_1, \sin\theta_1, 0, 0, 0, 0)$

$f_2 : (0, 2\pi) \to \mathbb{R}^6$, $\theta_2 \mapsto (0, 0, \cos\theta_2, \sin\theta_2, 0, 0)$ You don't need this.

$f_3 : (0, 2\pi) \to \mathbb{R}^6$, $\theta_3 \mapsto (0, 0, 0, 0, \cos\theta_3, \sin\theta_3)$

We attach 3 2-cells by the following mappings:

$g_1 : (0, 2\pi)^2 \to \mathbb{R}^6$ $(\theta_1, \theta_2) \mapsto (\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2, 0, 0)$

$g_2 : (0, 2\pi)^2 \to \mathbb{R}^6$ $(\theta_2, \theta_3) \mapsto (0, 0, \cos\theta_2, \sin\theta_2, \cos\theta_3, \sin\theta_3)$

$g_3 : (0, 2\pi)^2 \to \mathbb{R}^6$ $(\theta_1, \theta_3) \mapsto (\cos\theta_1, \sin\theta_1, 0, 0, \cos\theta_3, \sin\theta_3)$

Finally we attach the 3-cell:

$h : (0, 2\pi)^3 \to \mathbb{R}^6$ $(\theta_1, \theta_2, \theta_3) \mapsto (\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2, \cos\theta_3, \sin\theta_3)$

✓ Great

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