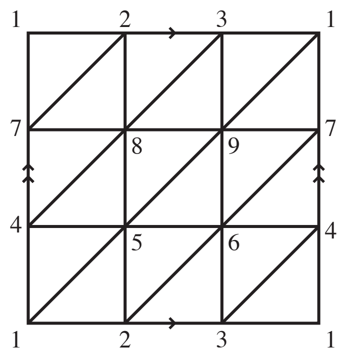
Topology & Groups

Michaelmas 2016

Question Sheet 4

1. Let K be a simplicial complex, and let α_1 and α_2 be edge paths. Suppose that α_1 and α_2 are homotopic relative to their endpoints. Show that α_1 and α_2 are equivalent as edge paths. [You should adapt the proof of Theorem III.27.]

2. Triangulate the torus as shown below



Let x and y be the loops $(1, 2, 3, 1)$ and $(1, 4, 7, 1)$, and let K be the union of these two loops (ie. K comes from the boundary of the square).

(i) Show that any edge path that starts and ends on K but with the remainder of the path missing K is equivalent to an edge path lying entirely in K.

(ii) Prove that any edge loop based at 1 is equivalent to an edge loop lying entirely in K.

(iii) Deduce that any edge loop based at 1 is equivalent to a word in the alphabet $\{x, y\}$.

(iv) Show that the edge loops xy and yx are equivalent.

(v) Deduce that any edge loop based at 1 is equivalent to $x^m y^n$, for $m, n \in \mathbb{Z}$.

(vi) Prove that if $x^m y^n \sim x^M y^N$, then $m = M$ and $n = N$. [Hint: define 'winding numbers' as in the proof of Theorem III.32.]

(vii) Deduce that the fundamental group of the torus is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

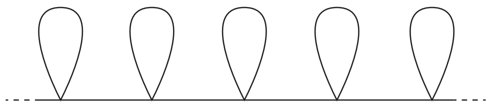
2. Prove that every non-trivial element of a free group has infinite order.

3. The centre $Z(G)$ of a group G is $\{g \in G : gh = hg \ \forall h \in G\}$. Let S be a set with more than one element. Prove that the centre of $F(S)$ is the identity element.

4. (i) Let F be the free group on the three generators x, y and z. For non-zero integers r, s and t, show that the subgroup of F generated by x^r, y^s and z^t is freely generated by these elements.

   (ii) Let H be the subgroup of $F(\{x, y\})$ generated by x^2, y^2, xy and yx. Show that H is not freely generated by these elements.

5. Compute an explicit free generating set for the fundamental group of the following graph:



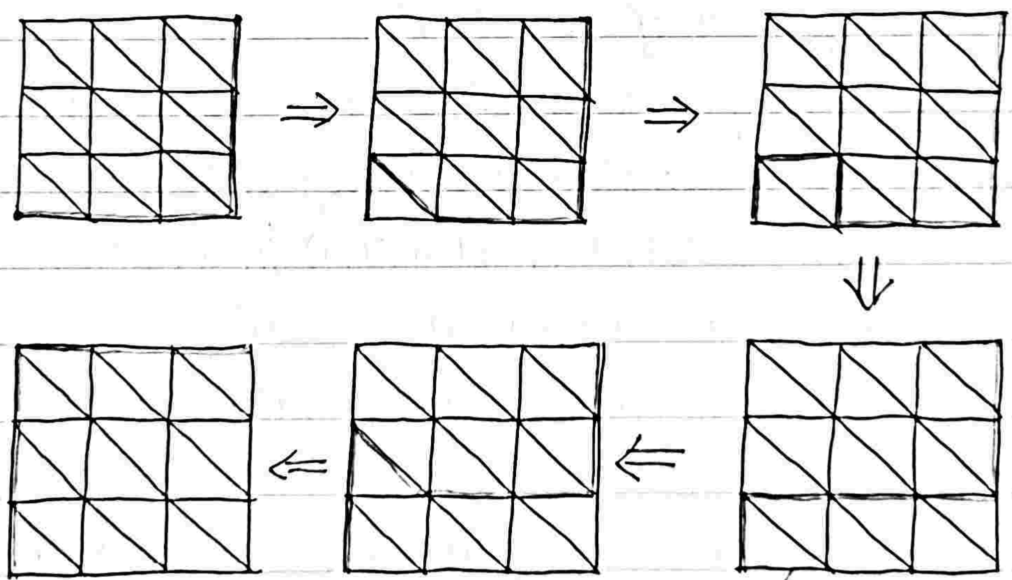
Topology & Groups 4 Peize Liu

1. Suppose that $\alpha_1 = (a_0, \cdots, a_n)$ and $\alpha_2 = (a_0', \cdots, a_n')$ where $a_0 = a_0'$, $a_n = a_n'$.

Since α_1 and α_2 are path-homotopic, the concatenation $\alpha_1 \cdot \alpha_2^{-1}$ is null-homotopic. $\Rightarrow [\alpha_1 \cdot \alpha_2^{-1}] = id_{\pi_1(|K|, a)}$

Since $E(K, a) \cong \pi_1(|K|, a)$, we have $[\alpha_1 \cdot \alpha_2^{-1}] = id_{E(K,a)}$ as an edge loop class. Hence $\alpha_1 \sim \alpha_2$.

Otherwise, suppose $\alpha_1 \simeq \alpha_2$ via a homotopy $H: I \times I \to |K|$. We can triangulate $I \times I$ and use the simplicial approximation theorem, which implies that there exists a simplicial map $G: (I \times I)_{(r)} \to K$ with $|G| \simeq H$. Moreover, $\alpha_1 = G|_{I \times \{0\}}$ and $\alpha_2 = G|_{I \times \{1\}}$. We can apply a sequence of elementary contractions and expansions taking α_1 to α_2:



A

Hence $\alpha_1 \sim \alpha_2$.

You should explain why the first and last pictures are α_1 and α_2.

$\boxed{\text{use straight line homotopy + Q1}}$

2. (i) It is clear from the diagram that any edge path can be taken to the sides of the square through a finite sequence of elementary contractions and expansions. For example, for

This is exactly what you need to prove

an edge path (1, 5, 8, 9, 7, 6, 3). We have the following sequence of elementary operations:

(1, 5, 8, 9, 7, 6, 3) ~ (1, 5, 9, 7, 6, 3) ~ (1, 5, 9, 6, 3)

~ (1, 5, 6, 3) ~ (1, 2, 5, 6, 3) ~ (1, 2, 6, 3) ~ (1, 2, 3)

which lies entirely in K.

We can do similar operations to any edge path.

(ii) ~~By (i),~~ Any edge loop is an edge path. So by (i) any edge loop based at 1 is equivalent to an edge path lying entirely in K. Since equivalent edge paths have the same starting and ending points, the edge path in K is still an edge loop based at 1. (i) has another condition.

(iii) Let x be the ~~equivalent~~ edge loop class of (1, 2, 3, 1) and y the edge loop class of (1, 4, 7, 1).

By (ii), any edge loop based at 1 is equivalent to an edge loop based at 1 ~~and~~ which lies entirely in K. Any loop in K must traverse (1, 2, 3, 1) or (1, 4, 7, 1) in a certain order. So it can be represented by a word in {x, y}. Could have 1, 2, 1, ...

(iv) It suffices to show that (1, 2, 3, 1, 4, 7, 1) and (1, 4, 7, 1, 2, 3, 1) are equivalent:

(1, 2, 3, 1, 4, 7, 1) ~ (1, 2, 3, 4, 7, 1) ~ (1, 2, 3, 6, 4, 7, 1)

~ (1, 2, 6, 4, 7, 1) ~ (1, 2, 6, 7, 1) ~ (1, 2, 5, 6, 7, 1)

~ (1, 2, 5, 6, 9, 7, 1) ~ (1, 5, 6, 9, 7, 1) ~ (1, 5, 9, 7, 1)

~ (1, 5, 9, 1) ~ (1, 4, 5, 9, 1) ~ (1, 4, 5, 8, 9, 1) ~

(1, 4, 5, 8, 9, 3, 1) ~ (1, 4, 8, 9, 3, 1) ~ (1, 4, 8, 3, 1) ~

(1, 4, 7, 8, 3, 1) ~ (1, 4, 7, 8, 2, 3, 1) ~ (1, 4, 7, 2, 3, 1)

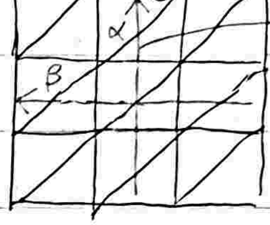
$\sim (1,4,7,1,2,3,1)$. ~~Vertices~~ * picture would be clearer.

Hence $xy = yx$.

(v) Any word in $\{x, y\}$ is a finite concatenation of ~~finit~~ x, x^{-1}, y, y^{-1}.
~~Since~~ We should have $xx^{-1} = x^{-1}x = e$ and $yy^{-1} = y^{-1}y = e$.
Since $xy = yx$, we have:

$$x^{-1}y^{-1} = y^{-1}x^{-1} \; ; \; xy^{-1} = y^{-1}x \; ; \; x^{-1}y = yx^{-1}.$$

That is, x, x^{-1}, y, y^{-1} all commute. Hence any word (i.e. edge loop based at 1) is equal to:



algebraic intersection number with loop α, β

$$\underbrace{x \cdots x}_{m'} \cdot \underbrace{x^{-1} \cdots x^{-1}}_{m''} \cdot \underbrace{y \cdots y}_{n'} \cdot \underbrace{y^{-1} \cdots y^{-1}}_{n''}$$

which is equivalent to $x^m y^n$ where $m = m' - m'' \in \mathbb{Z}$ and $n = n' - n'' \in \mathbb{Z}$.

(vi) Let the winding number $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ be defined as the previous part: $m = m' - m''$, $n = n' - n''$. We see that they are invariant under elementary operations. Hence $x^m y^n \sim x^M y^N \iff m = M$ and $n = N$. This is what you need to prove

(vii) The edge loop group $E(T^2, 1) \cong \mathbb{Z} \times \mathbb{Z}$ via the group isomorphism $\sigma : E(T^2, 1) \cong \mathbb{Z} \times \mathbb{Z}$ given by $\sigma(x^m y^n) = (m, n)$.
The injectivity of σ follows from part (vi). The surjectivity of σ is trivial, and so is the homomorphism property of σ.
By Theorem 3.27, since T^2 is path-connected, we have:
$E(T^2, 1) \cong \pi_1(T^2)$. Hence $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$. C†

3. Let $g \in F$ be a ~~non-element~~ non-trivial element of the free group F. By Proposition 4.8 g is uniquely represented by

a reduced word w.
Let z be the maximal subword of w such that $w = zxz^{-1}$.
(z could be $\emptyset$). Since w is non-trivial, we have $x \neq e$.
Now $g^2 = [ww] = [zxz^{-1}zxz] = [zxxz^{-1}]$ and inductively we have $g^n = [zx^n z^{-1}]$ for all $n \in \mathbb{N}$.
Notice that $\text{length}([xx]) = 2\,\text{length}([x])$. If this is not the case, then xx has cancellation and contradicts the maximality of z. Hence $\text{length}([x^n]) = n \cdot \text{length}([x])$.
But if $\exists\, n \in \mathbb{N} : g^n = e$, then $[x^n] = \emptyset$, which is a contradiction.
Therefore g must have infinite order. 4+

4. Suppose there is a non-trivial element $g \in Z(G)$, which is represented by a reduced word w.
~~Let z be the maximal subword of w such that $w = zxz$.~~
~~(z and x could be $\emptyset$)~~
Suppose $w = s_1^{e_1} \cdots s_n^{e_n}$ where $s_1, \dots, s_n \in S$ and $e_1, \dots, e_n \in \{-1, 1\}$.
If $s_1 = s_n$, then we choose $s_0 \in S \setminus \{s_1\}$ and let $h = [s_0 s_1^{-e_1}]$.
Since $gh = hg$, we have $[s_1^{e_1} \cdots s_n^{e_n} s_0 s_1^{-e_1}] = [s_0 s_2^{e_2} \cdots s_n^{e_n}]$.
~~LHS and RHS are both reduced words, but their lengths are different, which is a contradiction.~~
~~If $s_1 \neq s_n$, let $h = [$~~
LHS is a reduced word with length $n+2$, but RHS is a word with length n, which is a contradiction.
If $s_1 \neq s_n$, let $h = [s_n^{-e_n} s_1^{-e_1}]$. Since $gh = hg$, we have
$[s_n^{-e_n} s_2^{e_2} \cdots s_n^{e_n}] = [s_1^{e_1} s_2^{e_2} \cdots s_{n-1}^{e_{n-1}} s_1^{-e_1}] \Rightarrow [s_1^{e_1}] = [s_n^{-e_n}]$

$\Rightarrow S_1 = S_n$, which is a contradiction.

Hence $Z(G) = \{e\}$.

H +

5.(i) We claim that $\langle x^r, y^s, z^t \rangle = F(\{x^r, y^s, z^t\})$

This is quite obvious as x^r, y^s, z^t do not have any non-trivial relations. So $\langle x^r, y^s, z^t\rangle \cong F(\{x^r, y^s, z^t\})/\langle\langle e \rangle\rangle$

This is what you need to prove

$\cong F(\{x^r, y^s, z^t\})$

(ii) $F(\{x, y\})$ is not freely generated by x^2, y^2, xy and yx, as they ~~permit~~ have non-trivial relations: $(x^2)(yx)^{-1}(y^2)(xy)^{-1} = \emptyset$.

C +

6. The maximal tree of the graph is as follows:

Adding any edge would add an edge loop to this graph.

The remaining edges $E(\Gamma) \setminus E(T)$ are edge loops:

$\{e = (n, n) : n \in \mathbb{Z}\}$. (We can write $V(\Gamma)$ as $\mathbb{Z}$).

We choose a base vertex $0 \in V(\Gamma)$. By Theorem 4.11, π_1 of the graph is freely generately by

$\{\theta(n) \cdot (n, n) \cdot \theta(n)^{-1} : n \in \mathbb{Z}\}$

where $\theta(n)$ is the unique edge path from 0 to n.

We can see that this set is equipotent with $\mathbb{Z}$. The fundamental group is isomorphic to $F(\mathbb{Z})$.

Great

A +

KOKUYO