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Problem Sheet 1

Conformal Field Theory

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Question 1

For each of the following theories compute the canonical scaling dimensions of the fields and write down all the relevant and marginal operators (Remember: the operators should not have free Lorentz indices). Furthermore, give the scaling dimension of the coupling constants appearing in the Lagrangian.

Two scalar fields in $d = 3$ and $d = 4$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 + \frac{\lambda}{4!}\left(\phi_1^4 + \phi_2^4\right) + \frac{2\rho}{4!}\phi_1^2\phi_2^2$$

Dirac Lagrangian in $d = 4$

$$\mathcal{L} = \overline{\psi}\mathrm{i}\Gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi$$

Coleman–Weinberg model in $d = 4$

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + (D_\mu\phi)^2 + m^2\phi^2 + \lambda\phi^4$$

where the field ϕ is real, the signature is Euclidean and $D_\mu\phi = (\partial_\mu + eA_\mu)\phi$.

Proof. The scaling dimensions can be computed by a naive dimensional analysis. First we put everything into the natural units: $c = \hbar = 1$. Then let $[A]$ denote the *mass dimension* of the physical quantity A. Then we have:

$$[m] = 1, \quad [x] = -1, \quad [\partial_\mu] = 1, \quad [\mathcal{L}] = d, \quad [\phi] = \frac{d-2}{2}, \quad [\psi] = \frac{d-1}{2}.$$

An operator is relevant if the coupling has positive mass dimension, and is marginal if the coupling is dimensionless.

- **Two scalar fields in $d = 3$ and $d = 4$:**

  For $d = 4$, $[\phi] = 1$. The relevant operators are given by ϕ_1^2, ϕ_2^2, with coupling scaling dimension -2. The marginal operators are given by $(\partial_\mu\phi_1)^2, (\partial_\mu\phi_2)^2, (\partial_\mu\phi_1)(\partial^\mu\phi_2), \phi_1^4, \phi_2^4, \phi_1^2\phi_2^2$. In particular, we have $[\lambda] = [\rho] = 0$.

  For $d = 3$, $[\phi] = \frac{1}{2}$. The relevant operators are given by ϕ_1^2, ϕ_2^2 with coupling scaling dimension -2, and $\phi_1^4, \phi_2^4, \phi_1^2\phi_2^2$ with coupling scaling dimension -1. The marginal operators are given by $(\partial_\mu\phi_1)^2, (\partial_\mu\phi_2)^2, (\partial_\mu\phi_1)(\partial^\mu\phi_2), \phi_1^6, \phi_2^6, \phi_1^4\phi_2^2, \phi_1^2\phi_2^4$. In particular, we have $[\lambda] = [\rho] = 1$.

- **Dirac Lagrangian in $d = 4$:**

  We have $[\psi] = \frac{3}{2}$. The relevant operators are $\overline{\psi}\psi, \overline{\psi}\partial\!\!\!/\psi$ with coupling scaling dimension -1. (This is a free field theory and has no coupling constants.)

- **Coleman-Weinberg model in $d = 4$:**

  (*I don't know about quantum electrodynamics...*) We have $[\phi] = 1$, $[e] = 0$ and $[\partial_\mu] = [A_\mu] = 1$. It seems that the only relevant operator is ϕ^2 with coupling scaling dimension -2. The marginal operators are $F^{\mu\nu}F_{\mu\nu}, (\partial_\mu\phi)^2, (A_\mu\phi)^2, (\partial_\mu\phi)(A^\mu\phi), \phi^4$. □

Question 2

The beta function describes the dependence of a coupling parameter g on the energy scale μ of a given physical process:

$$\beta(g) = \mu \frac{\partial g}{\partial \mu}$$

such that at fixed points of the RG flow beta functions should vanish. Consider a nonabelian gauge theory with gauge group $\mathrm{SU}(N_c)$ and N_f Dirac fermions in the fundamental representation of the gauge group. For large N_c, N_f its perturbative beta function is given by

$$\beta(g) = -\frac{1}{16\pi^2}\frac{1}{3}\left(11N_c - 2N_f\right) g^3 - \frac{1}{(16\pi^2)^2}\left(\frac{34}{3}N_c^2 - \frac{13}{3}N_f N_c\right) g^5 + \cdots$$

Determine for which region of the parameters N_f, N_c there are fixed point consistent with perturbation theory.

Proof. The equation $\beta(g) = 0$ at order g^5 is given by

$$g^2 = -\frac{16\pi^2(11N_c - 2N_f)}{N_c(34N_c - 13N_f)}.$$

The equation has real solution $g \neq 0$ if and only if

$$(11N_c - 2N_f)(34N_c - 13N_f) < 0.$$

Therefore the fixed point exists only if $\frac{34}{13}N_c < N_f < \frac{11}{2}N_c$. □

Question 3

Given the finite conformal transformations check explicitly that a special conformal transformation is equivalent to an inversion followed by a translation followed by another inversion. How are the parameters a^μ and b^μ related?

Proof. A special conformal transformation is given by

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + x^2}.$$

Let $I\colon x^\mu \mapsto x^\mu/x^2$ be the inversion and $T_a\colon x^\mu \mapsto x^\mu + a^\mu$ be translation by a. We have

$$\begin{aligned} x'^\mu &= \frac{I(x)^\mu I(x)^{-2} - b^\mu I(x)^{-2}}{1 - 2b\cdot I(x) I(x)^{-2} + b^2 I(x)^{-2}} \\ &= \frac{I(x)^\mu - b^\mu}{(I(x) - b)^2} \\ &= \frac{T_{-b}\circ I(x)^\mu}{(T_{-b}\circ I(x))^2} = I\circ T_{-b}\circ I(x). \end{aligned}$$

Hence we have $a^\mu = -b^\mu$. □

Question 4

From the action of the conformal generators acting on functions, verify all the commutation relations of the conformal algebra except those between two Lorentz rotations.

Proof. The Lie brackets of the generators of Poincaré algebra has been verified in different contexts. We repeat the calculation here. First we note the canonical commutation relations:

$$[x_\mu, x_\nu] = [\partial_\mu, \partial_\nu] = 0, \qquad [\partial_\mu, x_\nu] = \eta_{\mu\nu}.$$

So we have

$$\begin{aligned} [P_\mu, P_\nu] &= 0; \\ [P_\rho, L_{\mu\nu}] &= [\partial_\rho, x_\mu]\partial_\nu - [\partial_\sigma, x_\nu]\partial_\mu = \eta_{\rho\mu}\partial_\nu - \eta_{\rho\nu}\partial_\mu = -\mathrm{i}(\eta_{\rho\nu}P_\mu - \eta_{\rho\mu}P_\nu). \end{aligned}$$

Next we consider the dilations:

$$\begin{aligned} [D, D] &= 0; \\ [D, P_\mu] &= -[x^\nu\partial_\nu, \partial_\mu] = -\partial_\mu = \mathrm{i}P_\mu; \\ [D, L_{\mu\nu}] &= [x^\sigma\partial_\sigma, x_\mu\partial_\nu - x_\nu\partial_\mu] = 0 \quad \text{(since } L_{\mu\nu} \text{ is skew-symmetric.)} \end{aligned}$$

Finally consider the special conformal transformations:

$$\begin{aligned} [K_\mu, K_\nu] &= -\left[2x_\mu x^\nu\partial_\nu - x^2\partial_\mu, 2x_\rho x^\sigma\partial_\sigma - x^2\partial_\rho\right] \\ &= -\left[x^2\partial_\mu, x^2\partial_p\right] - 4\left[x_\mu x^\nu\partial_\nu, x_\rho x^\sigma\partial_\sigma\right] + 2\left[x_\mu x^\nu\partial_\nu, x^2\partial_\rho\right] + 2\left[x^2\partial_\mu, x_\rho x^\sigma\partial_\sigma\right] \\ &= -2x^2(x_\mu\partial_\rho - x_\rho\partial_\mu) - 0 + 2(x^2x_\mu\partial_\rho - x^2\eta_{\mu\rho}x^\nu\partial_\nu) - 2(x^2x_\rho\partial_\mu - x^2\eta_{\mu\rho}x^\nu\partial_\nu) \\ &= 0 \\ [K_\mu, P_\nu] &= -\left[2x_\mu x^\rho\partial_\rho - x^2\partial_\mu, \partial_\nu\right] \\ &= -2\left[x_\mu x^\rho, \partial_\nu\right]\partial_\rho + \left[x^2, \partial_\nu\right]\partial_\mu \\ &= 2\eta_{\mu\nu}x^\rho\partial_\rho + 2\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \\ &= 2\mathrm{i}\left(\eta_{\mu\nu}D - L_{\mu\nu}\right) \\ [K_\mu, D] &= -\left[2x_\mu x^\rho\partial_\rho - x^2\partial_\mu, x^\nu\partial_\nu\right] \\ &= x^\nu\left[K_\mu, P_\nu\right] - \left[2x_\mu x^\rho\partial_\rho - x^2\partial_\mu, x^\nu\right]\partial_\nu \\ &= 2x_\mu x^\nu\partial_\nu + 2x_\mu x^\nu\partial_\nu - 2x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu + x^2\partial_\mu \\ &= 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu \\ &= \mathrm{i}K_\mu \\ [K_\mu, L_{\nu\rho}] &= \left[K_\mu, x_p P_\nu - x_\nu P_\rho\right] \\ &= x_p\left[K_\mu, P_\nu\right] - x_\nu\left[K_\mu, P_\rho\right] + \left[K_\mu, x_\rho\right]P_\nu - \left[K_\mu, x_\nu\right]P_\rho \\ &= 2\mathrm{i}\left(\left(x_\rho\eta_{\mu\nu} - x_\nu\eta_{\mu\rho}\right)D - \left(x_\rho L_{\mu\nu} - x_\nu L_{\mu\rho}\right)\right) + \cdots \\ &= \mathrm{i}\left(\eta_{\mu\nu}K_\rho - \eta_{\mu\rho}K_\nu\right). \end{aligned}$$

□

Question 5

Quadratic Casimir. Consider the following quadratic operator

$$\mathcal{C}^{(2)} = L_{\mu\nu}L^{\mu\nu} + \alpha P_\mu K^\mu + \beta K_\mu P^\mu + \gamma D^2$$

and fix the coefficients α, β, γ such that it commutes with all the generators of the conformal algebra.

Proof. This question is a computation by brute force.

$$\begin{aligned}
[\mathcal{C}^{(2)}, L_{\rho\sigma}] &= [L_{\mu\nu}L^{\mu\nu}, L_{\rho\sigma}] + \alpha[P_\mu K^\mu, L_{\rho\sigma}] + \beta[K_\mu P^\mu, L_{\rho\sigma}] + \gamma[D^2, L_{\rho\sigma}] \\
&= -\mathrm{i}\left(L_{\mu\rho}\eta_{\nu\sigma} - L_{\mu\sigma}\eta_{\nu\rho} - L_{\nu\rho}\eta_{\mu\sigma} + L_{\nu\sigma}\eta_{\mu\rho}\right) L^{\mu\nu} - \mathrm{i}\left(L^\mu{}_\rho\eta^\nu{}_\sigma - L^\mu{}_\sigma\eta^\nu{}_\rho - L^\nu_\rho\eta^\mu{}_\sigma + L^\nu{}_\sigma\eta^\mu{}_\rho\right) L_{\mu\nu} \\
&\quad + \alpha\mathrm{i}(P_\sigma\eta_{\rho\mu} - P_\rho\eta_{\sigma\mu})K^\mu + \alpha\mathrm{i}(K_\sigma\eta_\rho{}^\mu - K_\rho\eta_\sigma{}^\mu)P_\mu \\
&\quad + \beta\mathrm{i}(K_\sigma\eta_{\rho\mu} - K_\rho\eta_{\sigma\mu})P^\mu + \beta\mathrm{i}(P_\sigma\eta_\rho{}^\mu - P_\rho\eta_\sigma{}^\mu)K_\mu \\
&= 0
\end{aligned}$$

The calculation for P_σ:

$$\begin{aligned}
[\mathcal{C}^{(2)}, P_\rho] &= [L_{\mu\nu}L^{\mu\nu}, P_\rho] + \alpha[P_\mu K^\mu, P_\rho] + \beta[K_\mu P^\mu, P_\rho] + \gamma[D^2, P_\rho] \\
&= -\mathrm{i}\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right) L^{\mu\nu} - \mathrm{i}\left(\eta^\mu{}_\rho P^\nu - \eta^\nu{}_\rho P^\mu\right) L_{\mu\nu} \\
&\quad + 2\alpha\mathrm{i}(D\eta^\mu{}_\rho - L^\mu{}_\rho)P_\mu + 2\beta\mathrm{i}(D\eta_{\mu\rho} - L_{\mu\rho})P^\mu + 2\gamma\mathrm{i}DP_\rho \\
&= \mathrm{i}(2\alpha+\gamma)DP_\rho + \mathrm{i}(2\beta+\gamma)P_\rho D + \mathrm{i}(2-2\alpha)P^\mu L_{\mu\rho} + \mathrm{i}(2-2\beta)L_{\mu\rho}P^\mu.
\end{aligned}$$

So $[\mathcal{C}^{(2)}, P_\rho] = 0$ if

$$2\alpha + \gamma = 2\beta + \gamma = 2 - 2\alpha = 2 - 2\beta = 0.$$

That is, we have $\alpha = \beta = 1$ and $\gamma = -2$. The calculation for K_σ is similar:

$$[\mathcal{C}^{(2)}, K_\rho] = \mathrm{i}(2\alpha+\gamma)DK_\rho + \mathrm{i}(2\beta+\gamma)K_\rho D + \mathrm{i}(2-2\alpha)K^\mu L_{\mu\rho} + \mathrm{i}(2-2\beta)L_{\mu\rho}K^\mu = 0.$$

The calculation for D:

$$\begin{aligned}
[\mathcal{C}^{(2)}, D] &= \alpha[P_\mu K^\mu, D] + \beta[K_\mu P^\mu, D] \\
&= -\alpha\mathrm{i}P_\mu K^\mu + \alpha\mathrm{i}K_\mu P^\mu - \beta\mathrm{i}K_\mu P^\mu + \beta\mathrm{i}P_\mu K^\mu \\
&= (-\alpha+\beta)\mathrm{i}P_\mu K^\mu + (\alpha-\beta)\mathrm{i}K_\mu P^\mu \\
&= 0.
\end{aligned}$$

□

Question 6

Following the method of the lectures, derive $[K_\mu, \phi_\alpha(x)]$ for a primary *scalar* operator.

Proof. Following the lecture, we consider the group element

$$\widehat{K}_\mu =:= \exp(-\mathrm{i}x^\nu P_\nu)K_\mu \exp(\mathrm{i}x^\nu P_\nu) = \exp(\mathrm{i}x^\nu \operatorname{ad} P_\nu)K_\mu,$$

where $\operatorname{ad} P_\nu = [P_\nu, -]$ is the left adjoint action of P_ν. Expanding the exponent:

$$\widehat{K}_\mu = K_\mu + \mathrm{i}x^\nu[P_\nu, K_\mu] - \frac{1}{2}x^\nu x^\sigma[P_\sigma, [P_\nu, K_\mu]] - \frac{\mathrm{i}}{3!}x^\nu x^\sigma x^\rho[P_\rho, [P_\sigma, [P_\nu, K_\mu]]],$$

and all higher powers of action by P vanish. We can compute term by term:

$$\begin{aligned}
[P_\nu, K_\mu] &= -2\mathrm{i}(D\eta_{\mu\nu} - L_{\mu\nu}) \\
[P_\sigma, [P_\nu, K_\mu]] &= -2\mathrm{i}\,(\eta_{\mu\nu}[P_\sigma, D] - [P_\sigma, L_{\mu\nu}]) \\
&= -2\,(\eta_{\mu\nu}P_\sigma + \eta_{\mu\sigma}P_\nu - \eta_{\nu\sigma}P_\mu) \\
[P_\rho, [P_\sigma, [P_\nu, K_\mu]]] &= 0.
\end{aligned}$$

Hence

$$\widehat{K}_\mu = K_\mu + 2x^\nu(D\eta_{\mu\nu} - L_{\mu\nu}) + x^\nu x^\sigma(\eta_{\mu\nu}P_\sigma + \eta_{\mu\sigma}P_\nu - \eta_{\nu\sigma}P_\mu).$$

For a primary scalar operator $\phi_\alpha(x)$, we know that:

$$[P_\mu, \phi_\alpha(x)] = \mathrm{i}\partial_\mu\phi_\alpha(x), \qquad [L_\mu\nu, \phi_\alpha(0)] = 0, \qquad [K_\mu, \phi_\alpha(0)] = 0.$$

Therefore

$$\begin{aligned}
[\widehat{K}_\mu, \phi_\alpha(0)] &= 2x_\mu[D, \phi_\alpha(0)] + x^\nu x^\sigma(\eta_{\mu\nu}[P_\sigma, \phi_\alpha(0)] + \eta_{\mu\sigma}[P_\nu, \phi_\alpha(0)] - \eta_{\nu\sigma}[P_\mu, \phi_\alpha(0)]) \\
&= 2\mathrm{i}\Delta x_\mu\phi_\alpha(0) + 2\mathrm{i}x_\mu x^\sigma\partial_\sigma\phi_\alpha(0) - \mathrm{i}x^2\partial_\mu\phi_\alpha(0)
\end{aligned}$$

Therefore we have

$$[K_\mu, \phi_\alpha(x)] = 2\mathrm{i}\Delta x_\mu\phi_\alpha(x) + 2\mathrm{i}x_\mu x^\sigma\partial_\sigma\phi_\alpha(x) - \mathrm{i}x^2\partial_\mu\phi_\alpha(x). \qquad \square$$