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Problem Sheet 1

B4.3: Distribution Theory

30 October, 2020

this is much appreciated !!

Personal Conventions: $\mathbb{N}$ *denotes the set of non-negative integers.* $\mathbb{Z}_+$ *denotes the set of positive integers.*

I attempted Question 1 to 5. Q5(i) is not fully solved.

Question 1

Define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Show that ϕ is C^∞, and deduce that

$$\psi(x) = \phi(2(1-x))\phi(2(1+x))$$

belongs to $\mathscr{D}(\mathbb{R})$. Does the restriction to $(-1,1)$, $\psi|_{(-1,1)}$, belong to $\mathscr{D}(-1,1)$? Calculate the Taylor series for ϕ about 0 (note: not for ψ). Does the series converge, and if so, then what is its sum?

Proof. First we shall prove by induction on n that

$$\phi^{(n)}(x) = \begin{cases} p(x^{-1})\,e^{-x^{-1}}, & x > 0 \\ 0, & x \leqslant 0 \end{cases}$$

I'm not a huge fan of this... Just use product and chain rule? (This is a stylist issue more than anything)

where $p \in \mathbb{Q}[x]$ and $\deg p = 2n$. For $n = 0$ it is true. Suppose that it is true for $\phi^{(n)}(x)$. Then for $x > 0$,

$$\phi^{(n+1)}(x) = \frac{d}{dx}\left(p(x^{-1})\,e^{-x^{-1}}\right) = -t^2 \frac{d}{dt}\left(p(t)\,e^{-t}\right) = -x^{-2}\left(p'(x^{-1}) - p(x^{-1})\right)e^{-x^{-1}}$$

So the induction is true. Then begin next part of argument on a new line.

Let $q(x^{-1}) = x^{-2}\left(p(x^{-1}) - p'(x^{-1})\right)$. Then $\deg q = \deg p + 2 = 2(n+1)$. For $x < 0$, it is clear that $\phi^{(n+1)}(x) = 0$. As $x \searrow 0$,

$$\lim_{x \searrow 0} q(x^{-1})\,e^{-x^{-1}} = \lim_{t \to +\infty} q(t)\,e^{-t} = 0$$

Why? You need to use Taylor expansion & prove explicitly.

Hence $\phi^{(n+1)}(x) \to 0$ as $x \to 0$. The derivative at $x = 0$:

$$\phi^{(n+1)}(0) = \lim_{x \searrow 0} \frac{\phi^{(n)}(x)}{x} = \lim_{t \to +\infty} t\,p(t)\,e^{-t} = 0 = \lim_{x \nearrow 0} \frac{\phi^{(n)}(x)}{x}$$

Hence $\phi^{(n+1)}(x)$ exists and is continuous on $\mathbb{R}$.

We deduce that $\phi \in C^\infty(\mathbb{R})$. (because ϕ is C^∞ on $(-\infty, 0) \cup (0, \infty)$).

For $\psi(x) = \phi(2(1-x))\phi(2(1+x))$, it is clear that $\psi \in C^\infty(\mathbb{R})$. Since $\phi(2(1-x)) = 0$ for $x \geqslant 1$ and $\phi(2(1+x)) = 0$ for $x \leqslant -1$, $\operatorname{supp} \psi \in [-1,1]$. Hence ψ has a compact support. We deduce that $\psi \in \mathscr{D}(\mathbb{R})$.

just need to write by the chain rule & product rule

Note that $\psi|_{(-1,1)}$ has support $(-1,1)$, which is not compact. Then $\psi|_{(-1,1)} \notin \mathscr{D}(-1,1)$.

We have shown that $\phi^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Therefore the Taylor series of ϕ at $x = 0$:

$$\sum_{n=0}^{\infty} \frac{1}{n!}\phi^{(n)}(0)\,x^n = 0$$

The sum converges not to $\phi(x)$ but to 0. It implies that ϕ is not analytic near $x = 0$. □

Question 2

In this question all functions are real-valued.

(a) Let K be a compact proper subset of the open interval (a,b). Show carefully that there exists $\rho \in \mathscr{D}(a,b)$ such that $0 \leq \rho \leq 1$ and $\rho = 1$ on K.

(b) Give an example of $\varphi, \psi \in \mathscr{D}(\mathbb{R})$ such that $\max(\varphi,\psi)$, $\min(\varphi,\psi)$ are not smooth compactly supported test functions. Here we define $\max(\varphi,\psi)(x) = \max\{\varphi(x), \psi(x)\}$ for each x and similarly for $\min(\varphi,\psi)$.

Next, let $u \in \mathscr{D}(a,b)$. Show that there exist $u_1, u_2 \in \mathscr{D}(a,b)$ with $u_1 \geq 0, u_2 \geq 0$ and $u = u_1 - u_2$.

(c) Generalize the last statement to n dimensions as follows. Let Ω be a nonempty open subset of $\mathbb{R}^n$ and $u \in \mathscr{D}(\Omega)$. Show that there exist $u_1, u_2 \in \mathscr{D}(\Omega)$ with $u_1 \geq 0$ and $u_2 \geq 0$ such that $u = u_1 - u_2$

(*Hint: You may for instance note that* $4u = (u+1)^2 - (u-1)^2$ *and if* v *is a cut-off function between the support of* u *and the boundary of* Ω, *then* $vu = u$.)

Proof. (a) This is a special case of Theorem 2.11.

As $K \subseteq (a,b)$, let $0 < \delta < \frac{1}{4}\min\{\inf K - a, b - \sup K\}$. Let $\tilde{K} := \{x \in (a,b) : \exists y \in K \; |x-y| \leqslant 2\delta\} \subseteq (a,b)$. Let

$$B(x) := \begin{cases} \exp\left(\frac{1}{x^2-1}\right), & |x| < 1 \\ 0, & |x| \geqslant 1 \end{cases}$$

how does this relate to the prev. question?

Let

$$\rho_\delta(x) := \frac{1}{\delta \int_{\mathbb{R}} B(x)\,\mathrm{d}x} B\left(\frac{x}{\delta}\right)$$

be the standard modifier in $\mathbb{R}$. We know that $\rho_\delta \in \mathscr{D}(\mathbb{R})$ and it is supported on $\overline{B}(0,\delta)$. Let

$$\varphi(x) := \rho_\delta * \mathbf{1}_{\tilde{K}} = \int_{\mathbb{R}} \rho_\delta(x-y)\mathbf{1}_{\tilde{K}}(y)\,\mathrm{d}y = \int_{\tilde{K}} \rho_\delta(x-y)\,\mathrm{d}y$$

The map $y \mapsto \rho_\delta(x-y)$ is supported on $\overline{B}(x,\delta)$. Hence φ is supported on

$$\{z \in \mathbb{R} : \exists x \in \tilde{K}\; \exists y \in B(x,\delta) : \; |y-z| \leqslant \delta\} \subseteq \{z \in \mathbb{R} : \exists x \in K \; |x-z| \leqslant 3\delta\} \subseteq (a,b)$$

I would refer to L.N. here as DCT isn't exactly the whole argument.

It is clear by Dominated Convergence Theorem that φ is smooth. Hence $\varphi \in \mathscr{D}(a,b)$. Since $\int_{\mathbb{R}} \rho_\delta = 1$ and $\rho_\delta \geqslant 0$, we have $0 \leqslant \varphi \leqslant 1$ on (a,b).

More details.

For $x \in K$, as $y \mapsto \rho_\delta(x-y)$ is supported on $B(x,\delta) \subseteq \tilde{K}$, we have

$$\varphi(x) = \int_{\tilde{K}} \rho_\delta(x-y)\,\mathrm{d}y = \int_{\mathbb{R}} \rho_\delta(y)\,\mathrm{d}y = 1$$

hence $\varphi = 1$ on K.

(b) Let $\varphi(x) = B(x)$ and $\psi(x) = B(x-1)$. Then $\phi, \psi \in \mathscr{D}(\mathbb{R})$. But $\max\{\phi,\psi\}, \min\{\phi,\psi\} \notin \mathscr{D}(\mathbb{R})$ because they are not differentiable at $x = 1/2$:

$$\varphi(1/2) = B(1/2) = B(-1/2) = \psi(1/2)$$

This isn't really a full proof that the max & min fns arent diff'ble! You need some further details.

So $\max\{\varphi,\psi\}(1/2) = \min\{\varphi,\psi\}(1/2)$. But $\varphi'(1/2) > 0$ and $\psi'(1/2) < 0$. So $x = 1/2$ is a local minimum of $\max\{\varphi,\psi\}$ and a local maximum of $\min\{\varphi,\psi\}$. If they are differentiable then by Fermat's Lemma the derivative at $x = 1/2$ should be 0.

For $u \in \mathscr{D}(a,b)$, let $v : (a,b) \to [0,1]$ be a cut-off function between $\partial(a,b)$ and $\operatorname{supp} u$. Then $u = vu = v(u+1)^2/4 - v(u-1)^2/4$. Let $u_1 = v(u+1)^2/4$ and $u_2 = v(u-1)^2/4$. Then $u = u_1 - u_2$, $u_1, u_2 \geqslant 0$ and $u_1, u_2 \in \mathscr{D}(a,b)$. why?

(c) We should generalize (a) to a compact subset K of an open set $\Omega \in \mathbb{R}^n$. The proof is essentially the same. We define

$$\delta < \frac{1}{4}\operatorname{dist}(K, \partial\Omega)$$

and

$$\rho_\delta(x) := \frac{1}{\delta^n \int_{\mathbb{R}} B(x)\,\mathrm{d}x} B\left(\frac{x}{\delta}\right)$$

you haven't actually defined φ

The result φ is a cut-off function between K and $\partial\Omega$. Let v be a cut-off function between $\operatorname{supp} u$ and $\partial\Omega$. Take $u_1 = v(u+1)^2/4$ and $u_2 = v(u-1)^2/4$. □

And why do u_1, u_2 satisfy the hypotheses?

Question 3

Let Ω be a nonempty and open subset of $\mathbb{R}^n$, $1 \le p < \infty$ and $f \in \mathrm{L}^p(\Omega)$. Show that for each $\varepsilon > 0$ there exists $g \in \mathscr{D}(\Omega)$ such that $\|f - g\|_p < \varepsilon$.

(*Hint: One approach is to do it in two steps. First choose an appropriate open subset $O \subset \Omega$ so that $h = f\mathbf{1}_O$ is a good* L^p *approximation of f. Then use a result from lectures.*)

Proof. By Lemma 2.9, $\mathrm{C}_c^0(\Omega)$ is dense in $\mathrm{L}^p(\Omega)$ so there exists a $h \in \mathrm{C}_c^0(\Omega)$ such that $\|f - h\|_p < \varepsilon/2$. By Proposition 2.7(iii), $\lim_{\delta \to 0} \|\rho_\delta * h - h\|_p = 0$. Hence there exists $\delta > 0$ such that $\|\rho_\delta * h - h\|_p < \varepsilon/2$. Hence

$$\|f - \rho_\delta * h\|_p \le \|f - h\|_p + \|\rho_\delta * h - h\|_p < \varepsilon$$

By Proposition 2.7(i), $\rho_\delta * h \in \mathrm{C}^\infty(\Omega)$. It is also compactly supported as h is compactly supported. Hence $\rho_\delta * h \in \mathscr{D}(\mathbb{R})$, which completes the proof. □

is this true $\forall \delta > 0$? You could do w. more details here.

Question 4

Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Show that if $f \in \mathrm{L}^p(\mathbb{R}), g \in \mathrm{L}^q(\mathbb{R})$, then $f * g \in \mathrm{C}(\mathbb{R})$. Next, show that if $p \in (1, \infty)$, then $f * g \in \mathrm{C}_0(\mathbb{R})$, that is, $f * g$ is continuous and $(f * g)(x) \to 0$ as $|x| \to \infty$. What happens when $p = 1$ and $q = \infty$?

Proof. Without loss of generality we assume that $p \ne \infty$.

- By the result in Question 3, there exists a sequence $\{f_n\}$ in $\mathrm{C}_c^0(\mathbb{R})$ such that $f_n \to f$ in L^p-norm. We claim that $f_n * g \to f * g$ uniformly. Indeed,

$$\begin{aligned}\|f_n * g - f * g\|_\infty &= \sup_{x \in \mathbb{R}} \left| \int_{\mathbb{R}} (f_n - f)(x - y) g(y)\, \mathrm{d}y \right| \\ &\le \|g\|_q \sup_{x \in \mathbb{R}} \left(\int_{\mathbb{R}} |(f_n - f)(x - y)|^p\, \mathrm{d}y \right)^{1/p} && \text{(Hölder's Inequality)} \\ &= \|f_n - f\|_p \|g\|_q \to 0\end{aligned}$$

  as $n \to \infty$. Hence we have the uniform convergence.

- For each f_n, we claim that $f_n * g$ is uniformly continuous on $\mathbb{R}$. Since f_n is continuous and compactly supported, by Heine-Cantor Theorem it is uniformly continuous on $\mathbb{R}$:

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall x_1, x_2 \in \mathbb{R}\, \big(|x_1 - x_2| < \delta \implies |f_n(x_1) - f_n(x_2)| < \varepsilon\big)$$

  For $x_1, x_2 \in \mathbb{R}$ with $|x_1 - x_2| < \delta$,

$$\begin{aligned}|(f_n * g)(x_1) - (f_n * g)(x_2)| &= \left| \int_{\mathbb{R}} \big(f_n(x_1 - y) - f_n(x_2 - y)\big) g(x)\, \mathrm{d}y \right| \\ &\le \|g\|_q \left(\int_{\mathbb{R}} |f_n(x_1 - y) - f_n(x_2 - y)|^p\, \mathrm{d}y \right)^{1/p} && \text{(Hölder's Inequality)} \\ &\le \varepsilon \|g\|_q\, m(\operatorname{supp} f_n)^{1/p}\end{aligned}$$

  where m is the standard Lebesgue measure on $\mathbb{R}$. Hence $f_n * g$ is uniformly continuous on $\mathbb{R}$.

- Since $f * g$ is the uniform limit of the sequence of continuous functions $\{f_n * g\}$, we deduce that $f * g$ is continuous on $\mathbb{R}$. Show details for this?

Now we consider $p, q \ne \infty$. Then there exists $\{f_n\}$ and $\{g_n\}$ in $\mathrm{C}_c^0(\mathbb{R})$ such that $f_n \to f$ in L^p-norm and $g_n \to g$ in L^q-norm. Then we have

$$\|f_n * g_n - f * g\|_\infty = \|(f_n - f) * g_n + f * (g_n - g)\|_\infty \le \|f_n - f\|_p \|g\|_q + \|f\|_p \|g_n - g\|_q \to 0$$

as $n \to \infty$. Note that $\{f_n * g_n\}$ is a sequence in $\mathrm{C}_c^0(\mathbb{R})$ (why). Then for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$, $|(f * g)(x) - (f_n * g_n)(x)| < \varepsilon$ for all $x \in \mathbb{R}$ (since both $f_n * g_n$ and $f * g$ are continuous, the essential supremum is the supremum). Hence for $x \notin \operatorname{supp} f_n$, $|(f*g)(x)| < \varepsilon$. We deduce that $(f*g)(x) \to 0$ as $|x| \to \infty$.

It is not the case when $p = 1$ and $q = \infty$. A trivial example will be $f(x) = \mathrm{e}^{-x^2} \in \mathrm{L}^1(\mathbb{R})$ and $g(x) = 1 \in \mathrm{L}^\infty(\mathbb{R})$. Then

$$(f*g)(x) = \int_{-\infty}^{+\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t = \sqrt{\pi}$$

So $f*g$ does not tend to 0 as $|x| \to \infty$. □

Question 5

In each of the following 3 cases decide whether or not u_j is a distribution:

$$\langle u_1, \varphi \rangle = \sum_{j=1}^{\infty} 2^{-j} \varphi^{(j)}(0), \quad \langle u_2, \varphi \rangle = \sum_{j=1}^{\infty} 2^{j} \varphi^{(j)}(j), \quad \langle u_3, \varphi \rangle = \varphi(0)^2$$

where $\varphi \in \mathscr{D}(\mathbb{R})$ is so that the expression makes sense.

Proof. 1. u_1 is not a distribution.

First we claim that there exists a compact set $K \subseteq \mathbb{R}$ and a sequence $\{\varphi_n\} \subseteq \mathscr{D}(K)$ such that

$$|\langle u_1, \varphi_n \rangle| = \left| \sum_{j=1}^{\infty} 2^{-j} \varphi_n^{(j)}(0) \right| > n \sum_{j=0}^{n} \sup\left\{ \left| \varphi_n^{(j)}(x) \right| : x \in K \right\}$$

We can make, for example, all derivatives $\varphi^{(j)}(0) > 0$, and make $\varphi^{(n+1)}(0)$ arbitrarily large while the first n derivatives remain bounded in the compact set K. I am not sure if it is possible, given that K is independent of n.

(*In the remark after Example 3.12, a criterion of distribution is given as follows: Suppose that* $\{x_j : j \in J\} \subseteq \mathbb{R}$ *has no limit points. Then* $\langle T, \varphi \rangle := \sum_{j \in J} \varphi^{(\alpha_j)}(x_j)$ *is a distribution of order* $\sup_{j \in J} \alpha_j$. *This test fails for* u_1 *because all derivatives are evaluated at* 0.)

Put $\lambda_n = \langle u_1, \varphi_n \rangle$ and $\psi_n = \varphi_n / \lambda_n$. Then

$$\sum_{j=0}^{n} \sup\left\{ \left| \psi_n^{(j)}(x) \right| : x \in K \right\} < \frac{1}{n}$$

and hence $\left| \psi_n^{(j)}(x) \right| < 1/n$ for all $x \in K$ and $j \leqslant n$. In particular $\psi_n \to 0$ in $\mathscr{D}(\mathbb{R})$. But $\langle u_1, \psi_n \rangle = 1$ does not converge to 0.

2. u_2 is a distribution.

It is clear that u_2 is a linear functional. Suppose that $\{\varphi_n\} \subseteq \mathscr{D}(\mathbb{R})$ and $\varphi \in \mathscr{D}(\mathbb{R})$ such that $\varphi_n \xrightarrow{\mathscr{D}} \varphi$ as $n \to \infty$. There exists $N \in \mathbb{N}$ such that $\operatorname{supp} \varphi_n, \operatorname{supp} \varphi \subseteq [-N, N]$. Hence for $j > N$, $\varphi_n^{(j)}(j) = 0$ and $\varphi^{(j)}(j) = 0$.

We have $\varphi_n^{(j)}(j) \to \varphi^{(j)}(j)$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \langle u_1, \varphi_n \rangle = \lim_{n\to\infty} \sum_{j=1}^{N} 2^j \varphi_n^{(j)}(j) = \sum_{j=1}^{N} 2^j \varphi^{(j)}(j) = \langle u_1, \varphi \rangle$$

We deduce that u_2 is a distribution.

3. u_3 is not a distribution. In fact it is not even a linear functional:

$$\langle u_3, 2\varphi \rangle = (2\varphi(0))^2 = 4\varphi(0)^2 = 4\langle u_3, \varphi \rangle \neq 2\langle u_3, \varphi \rangle$$

for $\varphi \in \mathscr{D}(\mathbb{R})$ where $\langle u_3, 2\varphi \rangle \neq 0$. □

I'm not convinced by this argument so it seems circular. Will go through the counter ex. in class.

that and assume