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Problem Sheet 1

ASO: Calculus of Variations

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Question 1

Find the extremals of the functionals (assume that y is prescribed at $x = a$ and $x = b$):

(a) $\displaystyle\int_a^b (y^2 - y'^2 - 2y\cos 2x)\,\mathrm{d}x$

(b) $\displaystyle\int_a^b \frac{y'^2}{x^3}\,\mathrm{d}x$

(c) $\displaystyle\int_a^b (y^2 + y'^2 - 2y\,\mathrm{e}^x)\,\mathrm{d}x$

Solution. (a) Let $\mathcal{L}(x, y, y') = y^2 - y'^2 - 2y\cos 2x$ and $S = \displaystyle\int_a^b \mathcal{L}\,\mathrm{d}x$. The minimizer y of the action S satisfies the Euler-Lagrange Equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathcal{L}}{\partial y'} - \frac{\partial \mathcal{L}}{\partial y} = 0 \quad \Longrightarrow \quad y'' + y = \cos 2x$$

The boundary conditions $y(a) = y(a)$ and $y(b) = y(b)$ are not inhomogeneous. We can make it homogeneous by considering $v(x) := \dfrac{y(b) - y(a)}{b - a}(x - a) + y(a)$. Then the linear function v satisfies $v(a) = y(a)$ and $v(b) = y(b)$. Let $\tilde{y} := y - v$. $\tilde{y}$ satisfies the second order ODE with homogeneous boundary conditions:

$$\tilde{y}'' + \tilde{y} = \cos 2x - v(x), \qquad \tilde{y}(a) = \tilde{y}(b) = 0$$

For the homogeneous part, we note that $y_1(x) = \sin(x - a)$ and $y_2(x) = \sin(x - b)$ are two linearly independent solutions such that $y_1(a) = y_2(b) = 0$. The Wronskian:

$$W(x) := y_1(x)y_2'(x) - y_1'(x)y_2(x) = \sin(x-a)\cos(x-b) - \cos(x-a)\sin(x-b) = \sin(b-a)$$

Let $f(x) := \cos 2x - v(x)$. By Section 1.7 in the notes of Differential Equations II, we can write the full solution as:

$$\begin{aligned}
\tilde{y}(x) &= \int_a^x \frac{f(\xi)y_1(\xi)y_2(x)}{W(\xi)}\,\mathrm{d}\xi + \int_x^b \frac{f(\xi)y_2(\xi)y_1(x)}{W(\xi)}\,\mathrm{d}\xi \\
&= \frac{1}{\sin(b-a)}\left(\sin(x-b)\int_a^x (\cos 2\xi - v(\xi))\sin(\xi - a)\,\mathrm{d}\xi + \sin(x-a)\int_x^b (\cos 2\xi - v(\xi))\sin(\xi - b)\,\mathrm{d}\xi\right)
\end{aligned}$$

in which:

$$\begin{aligned}
\int_a^x \cos 2\xi \sin(\xi - a)\,\mathrm{d}\xi &= \int_a^x \frac{1}{2}(\sin(3\xi - a) - \sin(\xi + a))\,\mathrm{d}\xi = \frac{1}{6}(\cos 2a - \cos(3x - a)) - \frac{1}{2}(\cos 2a - \cos(x + a)) \\
\int_x^b \cos 2\xi \sin(\xi - b)\,\mathrm{d}\xi &= \int_x^b \frac{1}{2}(\sin(3\xi - b) - \sin(\xi + b))\,\mathrm{d}\xi = \frac{1}{6}(\cos(3x - b) - \cos 2b) - \frac{1}{2}(\cos(x + b) - \cos 2b) \\
\int_a^x v(\xi)\sin(\xi - a)\,\mathrm{d}\xi &= y(a)\int_a^x \sin(\xi - a)\,\mathrm{d}\xi + \frac{y(b) - y(a)}{b - a}\int_a^x (\xi - a)\sin(\xi - a)\,\mathrm{d}\xi \\
&= y(a)\int_a^x \sin(\xi - a)\,\mathrm{d}\xi + \frac{y(b) - y(a)}{b - a}\left((\xi - a)\cos(\xi - a)\big|_x^a + \int_a^x \cos(\xi - a)\,\mathrm{d}\xi\right) \\
&= y(a)(1 - \cos(x - a)) + \frac{y(b) - y(a)}{b - a}(-(x - a)\cos(x - a) + \sin(x - a)) \\
&= -v(x)\cos(x - a) + y(a) + \frac{y(b) - y(a)}{b - a}\sin(x - a) \\
\int_x^b v(\xi)\sin(\xi - b)\,\mathrm{d}\xi &= y(a)\int_x^b \sin(\xi - b)\,\mathrm{d}\xi + \frac{y(b) - y(a)}{b - a}\int_x^b (\xi - a)\sin(\xi - b)\,\mathrm{d}\xi \\
&= y(a)\int_x^b \sin(\xi - b)\,\mathrm{d}\xi + \frac{y(b) - y(a)}{b - a}\left((\xi - a)\cos(\xi - b)\big|_b^x + \int_x^b \cos(\xi - b)\,\mathrm{d}\xi\right) \\
&= y(a)(\cos(x - b) - 1) + \frac{y(b) - y(a)}{b - a}((x - a)\cos(x - b) - (b - a) - \sin(x - b))
\end{aligned}$$

$$= v(x)\cos(x-b) - y(b) - \frac{y(b)-y(a)}{b-a}\sin(x-b)$$

Substitute these expression into $\tilde{y}$ and we obtain:

$$\begin{aligned}
\tilde{y}(x) = &\frac{\sin(x-b)}{\sin(b-a)}\left(\frac{1}{6}(\cos 2a - \cos(3x-a)) - \frac{1}{2}(\cos 2a - \cos(x+a)) + v(x)\cos(x-a) - y(a) - \frac{y(b)-y(a)}{b-a}\sin(x-a)\right) \\
&+ \frac{\sin(x-a)}{\sin(b-a)}\left(\frac{1}{6}(\cos(3x-b) - \cos 2b) - \frac{1}{2}(\cos(x+b) - \cos 2b) - v(x)\cos(x-b) + y(b) + \frac{y(b)-y(a)}{b-a}\sin(x-b)\right) \\
&= \frac{1}{\sin(b-a)}\left(-\frac{1}{3}(\cos 2a\sin(x-b) - \cos 2b\sin(x-a)) + \frac{1}{6}(\sin(x-a)\cos(3x-b) - \sin(x-b)\cos(3x-a))\right. \\
&\left. + \frac{1}{2}(\sin(x-a)\cos(x+b) - \sin(x-b)\cos(x+a)) - v(x)\sin(b-a) + (-y(a)\sin(x-b) + y(b)\sin(x-a))\right) \\
&= v(x) - \frac{1}{3}\cos 2x + \left(y(a) + \frac{1}{3}\cos 2a\right)\frac{\sin(b-x)}{\sin(b-a)} + \left(y(b) + \frac{1}{3}\cos 2b\right)\frac{\sin(x-a)}{\sin(b-a)}
\end{aligned}$$

In conclusion, the solution to the original problem is given by:

$$y(x) = \tilde{y}(x) + v(x) = -\frac{1}{3}\cos 2x + \left(y(a) + \frac{1}{3}\cos 2a\right)\frac{\sin(b-x)}{\sin(b-a)} + \left(y(b) + \frac{1}{3}\cos 2b\right)\frac{\sin(x-a)}{\sin(b-a)}$$

(b) Let $\mathcal{L}(x, y, y') = y'^2/x^3$ and $S = \int_a^b \mathcal{L}\,\mathrm{d}x$. The minimizer y of the action S satisfies the Euler-Lagrange Equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial y'} - \frac{\partial\mathcal{L}}{\partial y} = 0 \quad\Longrightarrow\quad \frac{2y''}{x^3} = 0 \quad\Longrightarrow\quad y'' = 0$$

Hence y is a linear function. y satisfies the boundary conditions $y(a) = y(a)$ and $y(b) = y(b)$. The full solution is given by

$$y(x) = \frac{y(b)-y(a)}{b-a}(x-a) + y(a)$$

(c) Let $\mathcal{L}(x, y, y') = y^2 + y'^2 - 2y\,\mathrm{e}^x$ and $S = \int_a^b \mathcal{L}\,\mathrm{d}x$. The minimizer y of the action S satisfies the Euler-Lagrange Equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial y'} - \frac{\partial\mathcal{L}}{\partial y} = 0 \quad\Longrightarrow\quad y'' - y = -\mathrm{e}^x$$

For the homogeneous part, the auxiliary equation:

$$\lambda^2 - 1 = 0 \quad\Longrightarrow\quad \lambda_{1,2} = \pm 1$$

Then we find that $y_1(x) = \sinh(x-a)$ and $y_2(x) = \sinh(b-x)$ are two linearly independent solutions such that $y_1(a) = y_2(b) = 0$.

For the particular solution, we try $y_p(x) = \alpha x\,\mathrm{e}^x$. Then

$$-\mathrm{e}^x = y_p'' - y_p = \alpha(x+2)\,\mathrm{e}^x - \alpha x\,\mathrm{e}^x = 2\alpha\,\mathrm{e}^x \quad\Longrightarrow\quad \alpha = -\frac{1}{2}$$

Hence the general solution is given by

$$y(x) = -\frac{1}{2}x\,\mathrm{e}^x + A\sinh(x-a) + B\sinh(b-x)$$

With the help of part (a), we can immediately write down the full solution:

$$y(x) = -\frac{1}{2}x\,\mathrm{e}^x + \left(y(a) + \frac{1}{2}a\,\mathrm{e}^a\right)\frac{\sinh(b-x)}{\sinh(b-a)} + \left(y(b) + \frac{1}{2}b\,\mathrm{e}^b\right)\frac{\sinh(x-a)}{\sinh(b-a)}$$

□

Question 2

Find the extremals of

(a) $\displaystyle\int_0^1 (y^2 + y' + y'^2)\,\mathrm{d}x$ subject to $y(0) = 0,\ y(1) = 1$

(b) $\displaystyle\int_0^1 \frac{y'^2}{x^3}\,\mathrm{d}x$ subject to $y(0) = 1,\ y(1) = 2$

(c) $\displaystyle\int_0^1 y'^2\,\mathrm{d}x + y(1)^2$ subject to $y(0) = 1$.

Solution. (a) Let $\mathcal{L}(x, y, y') = y^2 + y' + y'^2$ and $S = \displaystyle\int_a^b \mathcal{L}\,\mathrm{d}x$. The minimizer y of the action S satisfies the Euler-Lagrange Equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathcal{L}}{\partial y'} - \frac{\partial \mathcal{L}}{\partial y} = 0 \quad\Longrightarrow\quad 2y'' - 2y = 0 \quad\Longrightarrow\quad y'' - y = 0$$

The second order ODE is linear and homogeneous. The auxiliary equation:

$$\lambda^2 - 1 = 0 \quad\Longrightarrow\quad \lambda_{1,2} = \pm 1$$

Hence the general solution is given by

$$y(x) = A\,\mathrm{e}^{x} + B\,\mathrm{e}^{-x}$$

Subtituting the boundary conditions $y(0) = 0$ and $y(1) = 1$:

$$A + B = 0 \qquad\qquad A\,\mathrm{e} + B\,\mathrm{e}^{-1} = 1$$

Hence $A = \dfrac{1}{2\sinh 1}$ and $B = -\dfrac{1}{2\sinh 1}$. The full solution is given by $y(x) = \dfrac{\sinh x}{\sinh 1}$.

(b) This is a special case of Question 1.(b). The solution is given by $y(x) = \dfrac{2-1}{1-0}(x - 0) + 1 = x + 1$

(c) First we assume that $y(1) = \eta$ is prescribed. Let $\mathcal{L}_\eta(x, y_\eta, y'_\eta) = y_\eta'^2$ and $S_\eta = \displaystyle\int_a^b \mathcal{L}_\eta\,\mathrm{d}x$. Then the minimizer y_η of the action S_η satisfies the Euler-Lagrange Equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathcal{L}}{\partial y'_\eta} - \frac{\partial \mathcal{L}}{\partial y_\eta} = 0 \quad\Longrightarrow\quad y''_\eta = 0$$

The solution is linear function:

$$y_\eta(x) = \frac{y(1) - y(0)}{1 - 0}(x - 0) + 1 = (\eta - 1)x + 1$$

Then the original functional becomes a function of η:

$$\int_0^1 y_\eta'^2\,\mathrm{d}x + y(1)^2 = \int_0^1 (\eta - 1)^2\,\mathrm{d}x + \eta^2 = 2\eta^2 - 2\eta + 1$$

which is minimized when $\eta = \dfrac{1}{2}$. Hence the minimizer of the original functional is given by $y = -\dfrac{1}{2}x + 1$

□

Question 3

Show that the problem of finding extremals of

$$J[y] = \int_a^b F(x, y, y')\,\mathrm{d}x$$

among all twice continuously differentiable functions y for which $y(a)$ is prescribed, leads to the Euler equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial y}$$

and to the natural boundary condition

$$\left.\frac{\partial F}{\partial y'}\right|_{x=b} = 0.$$

Find the extremal of $\int_0^1 \left(\frac{1}{2}y'^2 + yy' + y' + y\right)\,\mathrm{d}x$ among all y with $y(0) = 1$.

Proof. Suppose that y is a minimizer of the functional $J[y]$ subject to the constraint $y(a) = c \in \mathbb{R}$. We consider the set B of all test functions $\eta : \mathbb{R} \to \mathbb{R}$ such that $\operatorname{supp}\eta \subseteq [a, b]$ and $\eta(a) = 0$. Note that for any $\alpha \in \mathbb{R}$ and $\eta \in B$, the function $y + \alpha\eta$ satisfies the same constraint as y. Therefore we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha} J[y + \alpha\eta]\right|_{\alpha=0} = 0$$

By chain rule, we expand this as:

$$\begin{aligned}
0 = \left.\frac{\mathrm{d}}{\mathrm{d}\alpha} J[y + \alpha\eta]\right|_{\alpha=0} &= \int_a^b \left.\frac{\partial}{\partial\alpha} F(x, y + \alpha\eta, y' + \alpha\eta')\right|_{\alpha=0} \mathrm{d}x = \int_a^b \left(\eta\frac{\partial F}{\partial y} + \eta'\frac{\partial F}{\partial y'}\right) \mathrm{d}x \\
&= \int_a^b \eta\frac{\partial F}{\partial y}\,\mathrm{d}x + \left.\eta\frac{\partial F}{\partial y'}\right|_{x=a}^{x=b} - \int_a^b \eta\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}\,\mathrm{d}x \\
&= \left.\eta\frac{\partial F}{\partial y'}\right|_{x=b} - \int_a^b \eta\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y}\right) \mathrm{d}x \qquad \text{(we used the constraint } \eta(a) = 0\text{)}
\end{aligned}$$

Since $\eta \in B$ is arbitrary, we infer that y must satisfy the Euler-Lagrange Equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$$

with the natural boundary condition

$$\left.\frac{\partial F}{\partial y'}\right|_{x=b} = 0$$

Let $F(x, y, y') = \frac{1}{2}y'^2 + yy' + y' + y$. By Euler-Lagrange Equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(y' + y + 1) - (y' + 1) = 0 \quad\Longrightarrow\quad y'' = -1$$

The general solution is $y(x) = -\frac{1}{2}x^2 + Ax + B$. The boundary conditions are:

$$\begin{aligned}
y(0) = 1 &\quad\Longrightarrow\quad B = 1 \\
\left.\frac{\partial F}{\partial y'}\right|_{x=1} = y'(1) + y(1) + 1 = 0 &\quad\Longrightarrow\quad -\frac{1}{2} + 2A + B = 0
\end{aligned}$$

Hence $A = -\frac{1}{4}$, $B = 1$. The solution is $y(x) = -\frac{1}{2}x^2 - \frac{1}{4}x + 1$. □

Question 4

Show that the Euler equation of the functional

$$\int_{x_0}^{x_1} F(x, y, y', y'')\,\mathrm{d}x$$

has the first integral $F_{y'} - \frac{\mathrm{d}}{\mathrm{d}x}F_{y''} = \text{const}$ if $F_y \equiv 0$ and the first integral $F - y'\left(F_{y'} - \frac{\mathrm{d}}{\mathrm{d}x}F_{y''}\right) - y''F_{y''} = \text{const}$ if $F_x \equiv 0$.

Proof. By Section 6.2 in the notes, the Euler-Lagrange Equation of this problem is given by

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial F}{\partial y''} = 0$$

If $\frac{\partial F}{\partial y} = 0$, then the E-L Equation becomes:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial F}{\partial y''} = 0$$

By integrating with respect to x we find the first integral:

$$\frac{\partial F}{\partial y'} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y''} = \text{const}$$

If $\frac{\partial F}{\partial x} = 0$, by chain rule we find

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'} + y'''\frac{\partial F}{\partial y''} = y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'} + y'''\frac{\partial F}{\partial y''}$$

Substitute it into the E-L Equation:

$$\begin{aligned}
y'\frac{\partial F}{\partial y} - y'\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} + y'\frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial F}{\partial y''} &= 0 \\
\implies \frac{\mathrm{d}F}{\mathrm{d}x} - y''\frac{\partial F}{\partial y'} - y'''\frac{\partial F}{\partial y''} - y'\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} + y'\frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial F}{\partial y''} &= 0 \\
\implies \frac{\mathrm{d}F}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial F}{\partial y'}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y''}\right) - y''\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y''} - y'''\frac{\partial F}{\partial y''} &= 0 \\
\implies \frac{\mathrm{d}F}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial F}{\partial y'}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y''}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(y''\frac{\partial F}{\partial y''}\right) &= 0
\end{aligned}$$

By integrating with respect to x we find the first integral:

$$F - y'\left(\frac{\partial F}{\partial y'} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y''}\right) - y''\frac{\partial F}{\partial y''} = \text{const}$$

□

Question 5

Find the extremal of the functional

$$\int_0^{\pi/2} \left(y'^2 + z'^2 + 2yz\right)\, \mathrm{d}x$$

subject to $y(0) = 0$, $y\left(\frac{\pi}{2}\right) = 1$, $z(0) = 0$, $z\left(\frac{\pi}{2}\right) = 1$.

Solution. Let $\mathcal{L}(x, y, y', z, z') = y'^2 + z'^2 + 2yz$. The "system" has two "degrees of freedom". The corresponding Euler-Lagrange Equations are:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathcal{L}}{\partial y'} - \frac{\partial \mathcal{L}}{\partial y} = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathcal{L}}{\partial z'} - \frac{\partial \mathcal{L}}{\partial z} = 0$$

which gives:

$$2y'' - 2z = 0 \qquad 2z'' - 2y = 0$$

In matrix form it becomes:

$$\begin{pmatrix} y'' \\ z'' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

In general, such equations can be decoupled by diagnolizing the coefficient matrix. But for this simple system, we can decouple them by observation:

$$z = y'' \implies z'' = y'''' \implies y'''' - y = 0$$

This is a fourth order linear ODE. The auxiliary equation:

$$\lambda^4 - 1 = 0 \quad \implies \quad \lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = \mathrm{i},\ \lambda_4 = -\mathrm{i}$$

The general solution of y is given by $y(x) = A_1 \cos x + A_2 \sin x + A_3\, \mathrm{e}^{x} + A_4\, \mathrm{e}^{-x}$.

Hence $z(x) = y''(x) = -A_1 \cos x - A_2 \sin x + A_3\, \mathrm{e}^{x} + A_4\, \mathrm{e}^{-x}$.

The boundary conditions $y(0) = 0$, $y\left(\frac{\pi}{2}\right) = 1$, $z(0) = 0$, $z\left(\frac{\pi}{2}\right) = 1$ imply that

$$A_1 + A_3 + A_4 = 0 \qquad A_2 + A_3\, \mathrm{e}^{\pi/2} + A_4\, \mathrm{e}^{-\pi/2} = 1 \qquad -A_1 + A_3 + A_4 = 0 \qquad -A_2 + A_3\, \mathrm{e}^{\pi/2} + A_4\, \mathrm{e}^{-\pi/2} = 1$$

Hence $A_1 = A_2 = 0,\ A_3 = -A_4 = \dfrac{1}{2\sinh(\pi/2)}$. The solution of y is given by $y(x) = \dfrac{\sinh x}{\sinh(\pi/2)}$ and z by $z(x) = \dfrac{\sinh x}{\sinh(\pi/2)}$. □