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**Problem Sheet 3**

# String Theory II

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# 1 Super-Yang–Mills in various dimensions

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Let $A_\mu$ be a 10d $\mathrm{U}(N)$ gauge field and $\lambda$ a 16 component Majorana–Weyl spinor (gaugino) in 10d.

[ *Recall: spinor costruction in the lectures 3/4. Majorana means we impose a reality condition* $\lambda^* = B\lambda$. ]

**Question 1.1**

Show that

$$\mathcal{L}_{\text{10d SYM}} = -\frac{1}{4g_{\text{YM}}^2}\operatorname{Tr} F_{\mu\nu}F^{\mu\nu} - \frac{\mathrm{i}}{2g_{\text{YM}}^2}\operatorname{Tr}\left(\bar{\lambda}\Gamma^\mu D_\mu \lambda\right)$$

is invariant under the supersymmetry transformations with the supersymmetry parameter $\epsilon$

$$\begin{aligned} \delta A_\mu &= -\mathrm{i}\bar{\epsilon}\Gamma_\mu\lambda \\ \delta\lambda &= \frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon \end{aligned}$$

*Proof.* For the Yang-Mills theory, the field and the covariant derivative is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}[A_\mu, A_\nu], \qquad D_\mu\lambda = \partial_\mu\lambda - \mathrm{i}[A_\mu, \lambda].$$

The supersymmetric transformation for the field is given by

$$\begin{aligned} \delta F_{\mu\nu} &= \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu - \mathrm{i}[\delta A_\mu, A_\nu] - \mathrm{i}[A_\mu, \delta A_\nu] \\ &= -\mathrm{i}\bar{\epsilon}\left(\Gamma_\nu\partial_\mu\lambda - \Gamma_\mu\partial_\nu\lambda - \mathrm{i}\Gamma_\mu[\lambda, A_\nu] - \mathrm{i}\Gamma_\nu[A_\mu,\lambda]\right) - \mathrm{i}\lambda\left(\Gamma_\nu\partial_\mu\bar{\epsilon} - \Gamma_\mu\partial_\nu\bar{\epsilon}\right) \\ &= -\mathrm{i}\bar{\epsilon}\left(\Gamma_\nu D_\mu\lambda - \Gamma_\mu D_\nu\lambda\right) - \mathrm{i}\lambda\left(\Gamma_\nu\partial_\mu\bar{\epsilon} - \Gamma_\mu\partial_\nu\bar{\epsilon}\right) \end{aligned}$$

Therefore the first term in the Lagrangian is transformed by

$$\delta\left(-\frac{1}{4g^2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu})\right) = \frac{\mathrm{i}\bar{\epsilon}}{2g^2}\operatorname{tr}(F^{\mu\nu}\left(\Gamma_\nu D_\mu\lambda - \Gamma_\mu D_\nu\lambda\right)) + \frac{\mathrm{i}}{2g^2}\lambda\operatorname{tr}(F^{\mu\nu}\left(\Gamma_\nu\partial_\mu\bar{\epsilon} - \Gamma_\mu\partial_\nu\bar{\epsilon}\right))$$

I stuck here and I regretted not taking AQFT and supersymmetry in the Hilary term... □

**Question 1.2**

By Kaluza–Klein reducing along $T^d$, and thererby decomposing

$$\mathrm{SO}(1,9) \to \mathrm{SO}(d) \times \mathrm{SO}(1, 9-d) \tag{3}$$

(i.e. splitting the indices $\mu$ into sets along $d$ compact dimensions and $10-d$ non-compact dimensions) determine the Lagrangian in $10-d$ dimensions and the supersymmetry transformations.

[ *Hint: you will have to decompose the spinors according to (3).* ]

## 2 Free Fermion Description of the Heterotic String

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Consider the Vertex operator algebra generated for $\mu = 0, \cdots, 9$ by

- $X^\mu(z, \overline{z})$ be free bosonic fields (non-chiral)
- $\lambda^A(z), A = 1, \cdots, 32$ be holomorphic world-sheet fermion fields with periodic boundary conditions.
- $\widetilde{\psi}^\mu(\overline{z})$ be anti-holomorphic world-sheet fermion fields.

The OPE algebra is thus

$$\begin{aligned} X^\mu(z,\overline{z})X^\nu(0,0) &\sim -\eta^{\mu\nu}\frac{\alpha'}{2}\ln|z|^2 \\ \lambda^A(z)\lambda^B(0) &\sim \delta^{AB}\frac{1}{z} \\ \widetilde{\psi}^\mu(\overline{z})\widetilde{\psi}^\nu(0) &\sim \eta^{\mu\nu}\frac{1}{\overline{z}} \end{aligned}$$

**Question 2.1**

Compute the OPEs $TT$ and $\overline{TT}$, where

$$\begin{aligned} T(z) &= -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{2}\lambda^A\partial\lambda^A \\ \overline{T}(z) &= -\frac{1}{\alpha'}\overline{\partial} X^\mu \overline{\partial} X_\mu - \frac{1}{2}\widetilde{\psi}^\mu\overline{\partial}\widetilde{\psi}_\mu \end{aligned}$$

*Proof.* It suffices to compute the most singular term in the OPEs (i.e. the central charge) since we already know the general form of the $TT$ OPE. ✓

The OPE of $\lambda\lambda$ gives the following expectation values:

$$\langle \lambda^A(z)\lambda^B(w)\rangle = \delta^{AB}\frac{1}{z-w}, \quad \langle \partial\lambda^A(z)\lambda^B(w)\rangle = -\delta^{AB}\frac{1}{(z-w)^2}, \quad \langle \partial\lambda^A(z)\partial\lambda^B(w)\rangle = -2\delta^{AB}\frac{1}{(z-w)^3}.$$

By Wick contractions, we have

$$\begin{aligned} {:}\lambda^A(z)\partial\lambda^A(z){:}{:}\lambda^B(w)\partial\lambda^B(w){:} \sim &-\langle \lambda^A(z)\lambda^B(w)\rangle\langle \partial\lambda^A(z)\partial\lambda^B(w)\rangle + \langle \lambda^A(z)\partial\lambda^B(w)\rangle\langle \partial\lambda^A(z)\lambda^B(w)\rangle \\ &+ \mathcal{O}((z-w)^{-2}) \\ =& \, 2\delta^{AB}\delta^{AB}\frac{1}{(z-w)^4} - \delta^{AB}\delta^{AB}\frac{1}{(z-w)^4} + \mathcal{O}((z-w)^{-2}) \\ =& \, \frac{32}{(z-w)^4} + \mathcal{O}((z-w)^{-2}), \end{aligned}$$

Should compute the whole OPE at least for the $\lambda$s

which implies that the $-\frac{1}{2}{:}\lambda^A\partial\lambda^A{:}$ term in $T$ has central charge $c_\lambda = \frac{1}{2}\delta^{AA} = 16$. ✓ The anti-holomorphic fermionic fields are similar. They provide the central charge $\overline{c}_{\overline{\psi}} = \frac{1}{2}\delta^\mu{}_\mu = 5$. On ther other hand, we know that the free bosonic fields have central charge $(c, \overline{c}) = (10, 10)$. The $TT$ and $\overline{TT}$ OPEs are given by

$$\begin{aligned} T(z)T(w) &= \frac{26/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{regular terms.} \\ \overline{T}(\overline{z})\overline{T}(\overline{w}) &= \frac{15/2}{(\overline{z}-\overline{w})^4} + \frac{2\overline{T}(\overline{w})}{(\overline{z}-\overline{w})^2} + \frac{\overline{\partial}\overline{T}(\overline{w})}{\overline{z}-\overline{w}} + \text{regular terms.} \end{aligned}$$ ✓

□

**Question 2.2**

Using bosonization of the left moving fermions $\lambda^A$ show that this CFT is equivalent to the 10d SO(32) heterotic string.

*Proof.* *I am not sure if I understand the question correctly. From the 32 fermionic fields, it should be clear that the system satisfies a* SO(32) *symmetry by acting on the worldsheet fermions. This is the viewpoint taken by Polchinski §11.2. If we wish to relate this to the description given in the lectures using 16 worldsheet bosons, then the tool is to use bosonisation (cf. BLT §11.4). I don't know whether this is a purely algebraic trick or this has some deep physical insight...*

Using complex basis for the fermionic fields:

$$\Lambda^{\pm a} := \frac{1}{\sqrt{2}}\left(\lambda^{2a-1} \pm \mathrm{i}\lambda^{2a}\right), \qquad a = 1, \dots, 16.$$

√

Consider the Cartan subalgebra currents

$$J^{a,-a}(z) := :\Lambda^a(z)\Lambda^{-a}(z):$$

√

We claim that $J^{a,-a}(z)$ has the same CFT properties as $\partial X^a(z)$, where $X^a$ is a left-moving bosonic field. To prove this we compute the OPE:

$$\begin{aligned} J^{a,-a}(z)J^{b,-b}(w) &= \frac{1}{4}\left(:(\lambda^{2a-1})^2(z)::(\lambda^{2b-1})^2(w): + :(\lambda^{2a})^2(z)::(\lambda^{2b})^2(w):\right) \\ &= \frac{1}{2}\left(\left\langle\lambda^{2a-1}(z)\lambda^{2b-1}(w)\right\rangle^2 + \left\langle\lambda^{2a}(z)\lambda^{2b}(w)\right\rangle^2\right) + \cdots \\ &= \delta^{ab}\frac{1}{(z-w)^2} + \cdots \\ &= \text{const}\cdot\partial X^a(z)\partial X^b(w) \end{aligned}$$

other terms?

□

**Question 2.3**

Construct in the fermionic description (i.e. using $\lambda^A$) the mass-less states, taking care of imposing a GSO projection on both left and right-moving sectors.

*Proof.* (*cf.* Polchinski §11.2 pp. 50–53.) □

**Question 2.4**

Choose now boundary conditions for 16 of the $\lambda^A$, which are anti-periodic. Show that these fields realize a level 1 SO(16) × SO(16) current algebra. Using bosonization, construct in terms of $\lambda^A$ the current algebra that realizes the 10d $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string.

[ *Hint: you will need to construct momentum VO that extend the manifest symmetry* SO(16) × SO(16) *to* $\mathrm{E}_8 \times \mathrm{E}_8$. ]

*Proof.* □