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Problem Sheet 3

C2.6: Introduction to Schemes

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Question 1. Mostly topology, but useful

i) (Warm-up lemma) Let X be a topological space. Check that $\forall$ topological subspace $Y \subseteq X$:

- Y is irreducible $\implies$ Y is connected
- Y is irreducible $\implies$ $\overline{Y}$ is irreducible
- Y is an irreducible component $\implies$ Y is closed and connected

[*Recall that* ***irreducible component*** *means irreducible and maximal with respect to inclusion.*]

ii) Suppose X has finitely many irreducible components X_i.

We say that "X_k can be reached from X_l" if $X_k \cap X_{i_1} \neq \varnothing, X_{i_1} \cap X_{i_2} \neq \varnothing, \ldots, X_{i_n} \cap X_l \neq \varnothing$ for some X_{i_r}

Prove that X is connected $\iff$ any irreducible component can be reached from any other.

iii) *A topological space is* ***Noetherian*** *if it satisfies the descending chain condition for closed sets: $C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots \implies C_N = C_{N+1} = \cdots$ for some N.*

Prove that a Noetherian topological space has finitely many irreducible components, each containing an open dense set $\neq \varnothing$.

iv) Prove that R is a Noetherian ring $\implies$ $\operatorname{Spec} R$ is a Noetherian topological space. (*so for a Noetherian scheme every affine open is Noetherian topological space*)

Check that the converse fails for $k\left[x_1, x_2, x_3, \cdots\right] / \left(x_1, x_2^2, x_3^3, \ldots\right)$.

v) Prove that X is a Noetherian topological space $\iff$ every topological subspace of X is quasi-compact (*so for a Noetherian scheme X all subspaces are quasi-compact, not just X.*)

vi) Prove that X is a Noetherian scheme $\implies$ X is a Noetherian topological space.

Proof. i)

- Suppose that Y is connected. Then $Y = Y_1 \sqcup Y_2$ for non-empty Y_1 and Y_2 clopen in Y. Then by definition Y is reducible.
- Suppose that $\overline{Y}$ is reducible. Then $\overline{Y} = Y_1 \cup Y_2$ for non-empty Y_1, Y_2 closed in $\overline{Y}$. Then $Y = (Y \cap Y_1) \cup (Y \cap Y_2)$. By the definition of subspace topology, both $Y \cap Y_1$ and $Y \cap Y_2$ are closed in Y. Moreover they are non-empty. Suppose that $Y \cap Y_1 = \varnothing$. Then $Y \subseteq Y_2$ and hence $\overline{Y} \subseteq Y_2 \subsetneq \overline{Y}$. This is a contradiction. We deduce that Y is reducible.
- If Y is irreducible, by the previous result $\overline{Y}$ is irreducible. In particular if Y is an irreducible component then it is closed. We have shown that it is conencted.

ii) If X is disconnected, then it is clear that an irreducible component cannot reach another irreducible component in another connected component.

Conversely, suppose that X_1 and X_2 are two irreducible components of X that cannot be reacher from each other. Let

$$U_i := \bigcup \{X \colon X \text{ irreducible component that can be reached from } X_i\}$$

We claim that $U_1 \cup U_2$ is not connected. This is clear, as U_i is non-empty and closed in X, and $U_1 \cap U_2 = \varnothing$. Hence X is disconnected.

iii) Suppose that X is a Noetherian topological space. If $\{X_i\}_{i \in \mathbb{N}}$ is an infinitely set of irreducible components of X, then

$$\underset{X_0 \cup X_1}{\underset{\|}{Y_1}} \supsetneq \underset{X_0 \cup X_1 \cup X_2}{\underset{\|}{Y_2}} \supsetneq Y_3 \supsetneq \cdots$$

hmm

is a strictly descending chain of closed subspaces of X, where $Y_k = \bigcup_{i=0}^{k} X_i$. It follows that X has finitely many irreducible components.

Let $X_1, ..., X_N$ be the irreducible components of X. Let $Z_1 := X \setminus (\bigcup_{n=2}^{N} X_n) \subseteq X_1$. Then Z_1 is open and non-empty. If Z_1 is not dense in X_1, then $X_1 = \overline{Z}_1 \cup (X_1 \setminus Z_1)$ is reducible, which is contradictory. Hence X_1 contains a non-empty open dense subset. It is similar for $X_2, ..., X_N$.

iv) Suppose that $\operatorname{Spec} R$ is not a Noetherian topological space. Then there is a strictly descending chain of closed subsets:

$$\mathbb{V}(\mathfrak{p}_1) \supsetneq \mathbb{V}(\mathfrak{p}_2) \supsetneq \mathbb{V}(\mathfrak{p}_3) \supsetneq \cdots$$

Then we have an ascending chain of ideals of R:

$$\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \mathfrak{p}_3 \subsetneq \cdots$$

Hence R is not Noetherian.

Let $R := k[x_1, x_2, x_3, ...]/\langle x_1, x_2^2, x_3^3, ...\rangle$. We note that R is not Noetherian, because it has a strictly ascending chain of ideals

$$\langle x_2 \rangle \subsetneq \langle x_2, x_3 \rangle \subsetneq \langle x_2, x_3, x_4 \rangle \subsetneq \cdots$$

However, we shall prove that $\operatorname{Spec} R$ is a Noetherian topological space. Suppose that $\mathfrak{p} \in \operatorname{Spec} R$. Note that x_n is nilpotent for all $n \in \mathbb{N}$. Then $x_n \in \operatorname{Nil}(R) \subseteq \mathfrak{p}$ for all $n \in \mathbb{N}$. Hence $\langle x_2, x_3, ...\rangle \subseteq \mathfrak{p}$. But $\langle x_2, x_3, ...\rangle$ is maximal. We deduce that $\mathfrak{p} = \langle x_2, x_3, ...\rangle$ and $\operatorname{Spec} R = \{\langle x_2, x_3, ...\rangle\}$. Since $\operatorname{Spec} R$ is a finite set, trivially it is a Noetherian topological space.

v) Suppose that X is not a Noetherian topological space. There is a strictly descending chain of closed subsets:

$$Y_1 \supsetneq Y_2 \supsetneq Y_3 \supsetneq \cdots$$

which corresponds to a strictly ascending chain og open subsets:

$$Y_1^c \subsetneq Y_2^c \subsetneq Y_3^c \subsetneq \cdots$$

Let $Y := \bigcup_{n=1}^{\infty} Y_n^c$. Then $\{Y_n^c\}_{n=1}^{\infty}$ is an open cover of Y with no finite subcover. Hence Y is not compact.

Conversely, suppose that X has a subspace Y which is not compact. Let $\{Y_i\}_{i \in I}$ be an open cover of Y with no finite subcover. We construct a sequence $\{Y_{i_n}\}_{n \in \mathbb{N}}$ inductively as follows. First pick arbitrary $i_0 \in I$. Given $\{Y_{i_0}, ..., Y_{i_k}\}$, since this does not cover Y, we can find $i_{k+1} \in I$ such that $Y_{i_{k+1}} \nsubseteq \bigcup_{j=1}^{k} Y_{i_j}$. So we have a strictly ascending chain

$$Y_{i_1} \subsetneq Y_{i_2} \subsetneq Y_{i_3} \subsetneq \cdots$$

Each Y_{i_n} is open in Y, so $Y_{i_n} = Y \cap X_n$ for some X_n open in X. Then

$$X_1 \subsetneq X_2 \subsetneq X_3 \subsetneq \cdots$$

and hence

$$X_1^c \supsetneq X_2^c \supsetneq X_3^c \supsetneq \cdots$$

This is a strictly descending chain of closed subsets. We deduce that X is not a Noetherian topological space.

vi) Suppose that X is a Noetherian scheme. Let $\{U_1, ..., U_m\}$ be an affine open cover of X. $U_i \cong \operatorname{Spec} R_i$ for some Noetherian ring R_i. By (iv), U_i is a Noetherian topological space. Let $Y \subseteq X$ be a subscheme. Then $Y \cap U_i \subseteq U_i$. By (v), $Y \cap U_i$ is compact. Then $Y = \bigcup_{i=1}^{m}(Y \cap U_i)$ is compact. It follows that every subspace of X is compact. By (v), X is a Noetherian topological space. $\square$

Question 2

i) Check that $\mathbb{A}_k^2 = \operatorname{Spec} k[x,y]$ is a variety (*k is an algbraically closed field*)

[*Recall that a **variety** is a scheme which is integral, separated, finite type over* $\operatorname{Spec} k$.]

ii) Show that the open subscheme $\mathbb{A}_k^2 \setminus \{0\}$ is a variety which is not affine.

[*Hint. You may assume as known that being "finitely generated as a k-algebra" is affine-local: see notes Sec 3.2.*]

iii) Show that a variety which is affine (*being the spectrum of a ring*) is an **affine variety**, i.e. isomorphic to an integral closed subscheme of $\mathbb{A}_k^n$ for some n.

iv) Prove that $(X, \mathcal{O}_X)$ is a variety $\implies$ X is a Noetherian scheme.

v) Glue two copies of $\mathbb{A}_k^1 = \operatorname{Spec} k[x]$ along the basic open set $\mathbb{A}_k^1 \setminus \{0\} = D_x = \operatorname{Spec} k\left[x, x^{-1}\right]$ by the isomorphism $\operatorname{Spec} k\left[s, s^{-1}\right] \cong \operatorname{Spec} k\left[t, t^{-1}\right]$ given by $s \mapsto t$. Show that the glued scheme is not separated. (*compare notes Sec 5.3.*)

[*Hint: "equiliser"*]

vi) Let $(X, \mathcal{O}_X)$ be a variety, and $Z \subseteq X$ is an irreducible subspace.

[*Remark. Irreducibility is not vital if we allow varieties to be reducible.*]

*In notes Sec 5.5 you find the definition of what it means tor Z to be **locally closed** subscheme of X and how we construct a canonical induced reduced scheme structure $\mathcal{O}_Z$.*

- Prove that Z is locally closed $\implies$ $(Z, \mathcal{O}_Z)$ variety. [*Hint. 2(iv), 1(vi), 1(v) may help.*]
- If you define $\mathcal{O}_Z$ as suggested in Sec 5.5 for $Z \subseteq X$ irreducible subspace, prove that $(Z, \mathcal{O}_Z)$ variety $\implies$ $Z \subseteq X$ is locally closed

  Suggestion. First reduce to affine case $Z = \operatorname{Spec} S, X = \operatorname{Spec} R$ by picking $\operatorname{Spec} R \subseteq X$ of type open $\cap$ closed. Now we want to find an open set in Z such that the generating global sections over k come from sections on open $\subseteq X$. At the end, you may need to check $\operatorname{Spec} S \cap \operatorname{Spec} R_f = \operatorname{Spec} S_f$ ($S_f = S \otimes_R R_f$ via $\varphi^\# : R \to S$)

Proof. i) Let us unwrap the definitions.

- $(X, \mathcal{O}_X)$ is an integral scheme if $\mathcal{O}_X(U)$ is an integral domain for all open $U \subseteq X$. In Question 3 of Sheet 2, we have proven that $\operatorname{Spec} R$ is an integral scheme if and only if R is an integral domain. Since $k[x,y]$ is an inetgral domain, $\mathbb{A}_k^2$ is an integral scheme.
- X is separated over k, if the canonical morphism $f : X \to \operatorname{Spec} k$ is separated, which means that the diagonal map $\Delta : X \to X \times_{\operatorname{Spec} k} X$ is a closed immersion. A closed immersion $f : X \to Y$ is a morphism which is an isomorphism onto a closed subscheme $Z \subseteq Y$. A closed subscheme $Z \subseteq Y$ is a closed subset such that $j_* \mathcal{O}_Z \cong \mathcal{O}_Y / J$ for some quasi-coherent sheaf of ideals J on Y. A sheaf of ideals J is quasi-coherent if J is exhibited as the kernel of $\mathcal{O}_Y \to j_* \mathcal{O}_Z$, where $j : Z \to Y$ is the inclusion. (for some closed Z?)

  In this case $X = \mathbb{A}_k^2$ is affine. So $\Delta : X \to X \times_{\operatorname{Spec} k} X$ is induced by the k-algebra homomorphism $\varphi : k[x,y] \otimes_k k[x,y] \to k$ given by $f \otimes g \mapsto fg$. φ is surjective with

  $$\ker \varphi = \langle f \otimes 1 - 1 \otimes f : f \in k[x,y] \rangle$$

  Then $\Delta_{X/k} = \operatorname{im} \operatorname{Spec} \varphi = \mathbb{V}(\ker \varphi) \subseteq X \times_{\operatorname{Spec} k} X$. As $\mathbb{V}(\ker \varphi)$ is a closed affine subset of the affine scheme $X \times_{\operatorname{Spec} k} X$, it is cannonically a closed subscheme, because the ideal sheaf $\mathcal{O}_{\ker \varphi}$ is quasi-coherent. Moreover, Δ is an isomorphism onto $\mathbb{V}(\ker \varphi)$. Hence $X = \mathbb{A}_k^2$ is separated

  just quote "affines are separated"

over k. *The same method shows that the morphism* $\operatorname{Spec}\alpha\colon \operatorname{Spec}R \to \operatorname{Spec}S$ *induced by the monomorphism* $\alpha : S \to R$ *is always separated.* (precisely (it's in the notes))

- X is of finite type over k, if the canonical morphism $f : X \to \operatorname{Spec}k$ is of finite type, which means that the morphism is both quasi-compact and locally of finite type. $f : X \to \operatorname{Spec}k$ is quasi-compact if the pre-images of all affine open sets are quasi-compact. $f : X \to \operatorname{Spec}k$ is locally of finite type if for all affine open $U \subseteq X$ and $V \subseteq \operatorname{Spec}k$ with $f(U) \subseteq V$, the ring homomorphism $f^{\#} : \mathcal{O}_{\operatorname{Spec}k}(V) \to \mathcal{O}_X(U)$ is of finite type. In the lectures we have seen that for f being locally of fintie type, it suffices to take any affine open cover.

  In the case $X = \mathbb{A}^2_k$, $f : \operatorname{Spec}X \to \operatorname{Spec}k$ is induced by the inclusion $\iota : k \hookrightarrow k[x, y]$. Note that $\operatorname{Spec}k$ is a singleton as a set, and $k[x, y]$ is quasi-compact, so f is trivially quasi-compact. Both X and $\operatorname{Spec}k$ are affine, and the map $\iota^{\#}$ on $\operatorname{Spec}k$ is exactly ι. We know that $k[x, y]$ is a finitely generated k-algebra. So X is finite type over k.

In summary, $\mathbb{A}^2_k$ is an integral, separated, fintie type scheme over k. This proves that $\mathbb{A}^2_k$ is a variety.

ii) We claim that an irreducible open subscheme Y of a variety X is also a variety.

- We have seen in the lectures that being a reduced ring is a stalk-local property. So an open subscheme of a reduced scheme is also reduced. Then an irreducible open subscheme of X is integral by Sheet 2.

- By a remark in the notes, an open subscheme of a separated scheme over k is also separated over k.

- Since X is of finite type over k, X is quasi-compact. Let $\{X_1, ..., X_n\}$ be an affine open cover of X. Let $Y_i := Y \cap X_i$. So Y_i is an open subscheme of the affine scheme $X_i \cong \operatorname{Spec}R_i$, where R_i is of finite type over k. Then Y_i has an open cover $\{D_{f_1}, ..., D_{f_m}\}$ for some $f_1, ..., f_m \in R_i$. Each $D_{f_j} \cong \operatorname{Spec}(R_i)_{f_j}$, where $(R_i)_{f_j}$ is of finite type over k. We have seen in the lectures that being a finitely generated k-algebra is an affine-local property. Therefore Y_i is locally of finite type. But also $Y = \bigcup_{i=1}^n Y_i$, so Y is also locally of finite type.

  Finally, Y is a finite union of some affine open subsets, which are quasi-compact. Therefore Y is also quasi-compact. Hence Y is of finite type over k.

This concludes the proof of the claim.

Since $\mathbb{A}^2_k \setminus \{0\}$ is an open subscheme of $\mathbb{A}^2_k$, it is a variety. We shall prove that $Y := \mathbb{A}^2_k \setminus \{0\}$ is not affine by proving that $\mathcal{O}_Y(Y) = k[x, y]$ (*which is in fact proven in C3.4 Algebraic Geometry*).

We note that $\mathbb{A}^2_k \setminus \{0\} = D_x \cup D_y$ for $x, y \in k[x, y]$. To see this, we simply have

$$\mathfrak{p} \in D_x \cup D_y \iff x \notin \mathfrak{p} \vee y \notin \mathfrak{p} \iff \mathfrak{p} \neq \langle x, y\rangle \iff \mathfrak{p} \in \mathbb{A}^2_k \setminus \{0\}.$$

We have $\mathcal{O}_{\mathbb{A}^2_k}(D_x) = k[x, y]_x = k[x, y, x^{-1}]$ and $\mathcal{O}_{\mathbb{A}^2_k}(D_y) = k[x, y]_y = k[x, y, y^{-1}]$. By uniqueness of the sheaf, we must have

$$\mathcal{O}_{\mathbb{A}^2_k\setminus\{0\}}(\mathbb{A}^2_k \setminus \{0\}) = \mathcal{O}_{\mathbb{A}^2_k}(\mathbb{A}^2_k \setminus \{0\}) = \mathcal{O}_{\mathbb{A}^2_k}(D_x) \cap \mathcal{O}_{\mathbb{A}^2_k}(D_y) = k[x, y]$$

If $\mathbb{A}^2_k \setminus \{0\}$ is affine, then $\mathbb{A}^2_k \setminus \{0\} \cong \operatorname{Spec}k[x, y] = \mathbb{A}^2_k$, which is impossible. Hence $\mathbb{A}^2_k \setminus \{0\}$ is not an affine variety.

justify

iii) Suppose that $X = \operatorname{Spec}R$ is a variety. By definition, we know that R is of finite type over k. There exists a surjection $\varphi : k[x_1, ..., x_n] \to R$. Then $\operatorname{Spec}\varphi : X \to \mathbb{A}^n_k$ is a closed immersion by definition. Hence X is isomorphic to an closed subscheme of $\mathbb{A}^n_k$. Since X is a variety, X is integral. We deduce that X is an affine variety.

iv) Since X is a variety, X is quasi-compact. So we only need to show that X is locally Noetherian, which is an affine-local property. For affine open set $U \subseteq X$ such that $U \cong \operatorname{Spec} R$, we know that R is a of finite type over k. We have a short exact sequence of k-algebras

$$0 \longrightarrow \ker \varphi \longrightarrow k[x_1, ..., x_n] \xrightarrow{\varphi} R \longrightarrow 0$$

Since $k[x_1, ..., x_n]$ is Noetherian by Hilbert basis theorem, so is R. We conclude that X is a Noetherian scheme.

v) Let $X := \mathbb{A}^1 \cup_{\mathbb{A}^1 \setminus \{0\}} \mathbb{A}^1$ be the glued scheme. Suppose that X is separated. We look at the two affine open sets U_1, U_2 in X isomorphic to $\mathbb{A}^1 = \operatorname{Spec} k[x]$, their intersection is $U_1 \cap U_2 \cong \mathbb{A}^1 \setminus \{0\} = \operatorname{Spec} k[x, x^{-1}]$. By Question 3.(iv) (or a claim in the notes), the multiplication map

$$m : \mathcal{O}_X(U_1) \otimes_k \mathcal{O}_X(U_2) \to \mathcal{O}_X(U_1 \cap U_2)$$

is surjective. In fact m is the k-algebra homomorphism $m : k[x] \otimes_k k[x] \to k[x, x^{-1}]$, which is clearly not surjective because $x^{-1} \notin \operatorname{im} m$. Hence X is not separated.

vi) • Suppose that Z is locally closed. We know that Z is open in $\overline{Z}$. We claim that the unique induced reduce subscheme structure on $\overline{Z} \subseteq X$ makes $\overline{Z}$ a subvariety of X. Then it follows from (ii) that Z is a variety.

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□

Question 3

Let $f : X \to B$ be a morphism of schemes.

i) *f is called an **immersion** (or locally closed immersion) if f is the composition $X \to U \to B$, where $X \to U$ is a closed immersion and $U \to B$ is an open immersion.*

Show that an immersion is a closed immersion $\iff$ $f(X) \subseteq B$ closed set.

[*Hint. For* $\Longleftarrow$ *: glue the ideal sheaf of* $X \xrightarrow{\varphi} U$ *with* $\mathcal{O}_X|_{B \setminus \varphi(X)}$, *and check the quasi-coherence.*]

ii) Show that $\Delta_{X/B} \subseteq X \times_B X$ is closed if B and X affine (*notation of notes Sec 5.3*)

iii) Show that $\Delta_{X/B}$ is an immersion.

[*Hence f is separated* $\iff$ $\Delta_{X/B}$ *is a closed immersion* $\iff$ $\Delta_{X/B}$ *is a closed set.*]

iv) *We say that $U, V \subseteq X$ are "nice" if $U, V, U \cap V$ are affine open sets and $\mathcal{O}_X(U) \otimes_{\mathbb{Z}} \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is surjective.*

- Suppose f is separated. Prove that for all affine open $U, V \subseteq X$ such that $f(U), f(V)$ are contained in an affine open subset of B, U, V are nice.
- Suppose that there exists an open cover $X = \bigcup U_i$ such that for all $x, y \in X$ with $f(x) = f(y)$, there are nice U_i, U_j with $x \in U_i$, $y \in U_j$ and $f(U_i), f(U_j)$ are subsets of an affine open set of B. Prove that f is separated.

[*For* $B = \operatorname{Spec} k$*:* ($\exists$ *open cover* $X = \bigcup U_i$, *all* U_i, U_j *nice*) $\Longrightarrow$ (f *separated*) $\Longrightarrow$ (*all affine opens* U, V *are nice*)]

v) Show that $\mathbb{P}^n_k$ is separated using (iv) (k any field). Deduce that $\mathbb{P}^n_k$ is a variety.

Show that **projective varieties** (*integral closed subschemes of* $\mathbb{P}^n_k$) and **quasi-projective varieties** (*irreducible open subschemes of a projective variety*) are varieties.

Proof. i) We propose the follow lemma:

> *A morphism $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a closed immersion if and only if f is a homeomorphism onto the closed subset $f(X) \subseteq Y$ and $f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is surjective for all $x \in X$.*

Given this lemma, the proof of (i) is straightforward. " $\Longrightarrow$ " is just the definition of a closed immersion. For " $\Longleftarrow$ ", since f is an immersion we have $f_x^\# = \psi_{\varphi(x)}^\# \circ \varphi_x^\#$, where $\psi_{\varphi(x)}^\# : \mathcal{O}_{B,f(x)} \to \mathcal{O}_{U,\varphi(x)}$ is an isomorphism, and $\varphi_x^\# : \mathcal{O}_{U,\varphi(x)} \to \mathcal{O}_{X,x}$ is a surjection. Hence $f_x^\#$ is surjective for all $x \in X$. By the lemma we deduce that f is a closed immersion.

ii) Suppose that $X \cong \operatorname{Spec} R$ and $B \cong \operatorname{Spec} A$ for some rings R and A. Then the map $\Delta : X \to X \times_B X$ is induced from the A-algebra homomorphism $\varphi : R \otimes_A R \to R$ given by $r \otimes s \mapsto rs$. So $\Delta = \operatorname{Spec} \varphi : \operatorname{Spec} R \to \operatorname{Spec}(R \otimes_A R)$. We claim that $\Delta_{X/B} = \mathbb{V}(\ker \varphi) = \mathbb{V}(\langle r \otimes 1 - 1 \otimes r : r \in R \rangle)$. This is immediate from that $R \cong (R \otimes_A R)/\ker \varphi$. So $\Delta_{X/B}$ is closed in $\operatorname{Spec}(R \otimes_A R) \cong X \times_B X$. *Moreover, the morphism Δ is a closed immersion.*

iii) Let $\{U_i\}_{i \in I}$ be an affine cover of X. (With possible refinement of this cover) for each U_i, let V_i be an affine open of B such that $f(U_i) \subseteq V_i$. Then we know that each $U_i \times_{V_i} U_i$ is affine open in $X \times_B X$. Let $Y := \bigcup_{i \in I} U_i \times_{V_i} U_i$. Then there is a canonical open immersion $Y \to X \times_B X$. It is clear that Δ maps X into Y. We need to show that this is a closed immersion. But by (ii) we already know that $\Delta_{U_i/V_i} : U_i \to U_i \times_{V_i} U_i$ is a closed immersion, and that $\{U_i \times_{V_i} U_i\}_{i \in I}$ is an affine open cover for Y. By the notes we deduce that $\Delta : X \to Y$ is a closed immersion. Hence $\Delta : X \to X \times_B X$ is an immersion.

iv)
- Let $f : X \to B$ be separated. Suppose that $U \cong \operatorname{Spec} R$ and $V \cong \operatorname{Spec} S$. Suppose that $f(U), f(V) \subseteq C$, where $C \cong \operatorname{Spec} A$ is affine open in B. Then $U \times_B V \cong U \times_C V$ is affine in $X \times_B X$. In particular, we have

$$\mathcal{O}_{X \times_B X}(U \times_B V) \cong \mathcal{O}_X(U) \otimes_A \mathcal{O}_X(V) \cong R \otimes_A S$$

  On the other hand, we note that $U \cap V = \Delta_{X/B}^{-1}(U \times_B V)$. Since $\Delta_{X/B}$ is a closed immersion, we have

$$U \cap V \cong \Delta_{X/B}(U \cap V) = \Delta_{X/B}(\Delta_{X/B}^{-1}(U \times_B V)) = \Delta_{X/B}(X) \cap (U \times_B V)$$

  Since $\Delta_{X/B}(X)$ is closed in $X \times_B X$, then $U \cap V$ is isomorphic to a closed subset of $U \times_B V$. Since $U \times_B V \cong \operatorname{Spec}(R \otimes_A S)$ is affine, $U \cap V$ is also affine, and we have $U \cap V \cong \operatorname{Spec}((R \otimes_A S)/I)$ for some ideal I of $R \otimes_A S$. In particular we have a surjective A-algebra homomorphism

$$\mathcal{O}_{X \times_B X}(U \times_B V) \to \mathcal{O}_X(U \cap V)$$

  Finally, since A is naturally a $\mathbb{Z}$-algebra (i.e. a ring), we have the canonical surjective ring homomorphism

$$\mathcal{O}_X(U) \otimes_{\mathbb{Z}} \mathcal{O}_X(V) \to \mathcal{O}_X(U) \otimes_A \mathcal{O}_X(V)$$

  Composing the maps above we obtain a surjective ring homomorphism

$$\mathcal{O}_X(U) \otimes_{\mathbb{Z}} \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$$

  Therefore U, V are nice.

- Since for each U_i and U_j there exists an affine open C of B such that $f(U_i), f(U_j) \subseteq C$, then $U_i \times_B U_j \cong U_i \times_C U_j$ is affine open in $X \times_B X$. $X \times_B X$ has an affine cover $\{U_i \times_B U_j\}_{i,j \in I}$ by the given assumption. Note that $\Delta_{X/B}^{-1}(U_i \times_B U_j) = U_i \cap U_j$. Since U_i, U_j are nice, we have a surjection

$$\mathcal{O}_X(U_i) \otimes_{\mathbb{Z}} \mathcal{O}_X(U_j) \to \mathcal{O}_X(U_i \cap U_j)$$

  If $C \cong \operatorname{Spec} A$, then the above map is A-bilinear, and hence factors through $\mathcal{O}_X(U_i) \otimes_A \mathcal{O}_X(U_j) \cong$

$\mathcal{O}_{X\times_B X}(U_i \times_B U_j)$. Hence we have a surjection $\mathcal{O}_{X\times_B X}(U_i \times_B U_j) \to \mathcal{O}_X(U_i \cap U_j)$. Hence $\Delta_{X/B} : U_i \cap U_j \to U_i \times_B U_j$ is a closed immersion. By the notes we conclude that $\Delta_{X/B} : X \to X \times_B X$ is a closed immersion.

v) Recall from Question 1 of Sheet 2 that $\mathbb{P}_k^n = \bigcup_{i=0}^n U_i$ where each $U_i \cong \mathbb{A}_k^n = \operatorname{Spec} k[x_1, ..., x_n]$. The pairwise intersection $U_i \cap U_j \cong \operatorname{Spec} R_{ij}$, where R_{ij} is the 0th grading of the ring of fractions $S^{-1}k[x_0, ..., x_n]$, S is the multiplcative set generated by x_i, x_j. Next we look at the multiplication homomorphism between the global sections of affine sets:

$$\varphi : \mathcal{O}_X(U_i) \otimes_k \mathcal{O}_X(U_j) \to \mathcal{O}_X(U_i \cap U_j)$$

Recall that $\mathcal{O}_X(U_i) = R_i = k\left[\frac{x_0}{x_i}, ..., \widehat{\frac{x_i}{x_i}}, ..., \frac{x_n}{x_i}\right]$ and $\mathcal{O}_X(U_i \cap U_j) = R_{ij} = R_i\left[\frac{x_i}{x_j}\right]$. Note that $x_i/x_j \in R_j$. Every element in R_{ij} takes the form $\sum_{m=0}^{\ell} a_m (x_i/x_j)^m$ for $a_m \in R_i$. Then

$$\sum_{m=0}^{\ell} a_m (x_i/x_j)^m = \varphi\left(\sum_{m=0}^{\ell} a_m \otimes (x_i/x_j)^m\right)$$

So φ is surjective. We deduce that $\{U_0, ..., U_n\}$ is an open cover of $\mathbb{P}_k^n$ which is pairwise "nice". Using the notation from (iv), $B = \operatorname{Spec} k$ is a singleton. The conditions on the nice affine open cover are satisfied trivially. Hence $\mathbb{P}_k^n$ is separated over k.

Checking the remaining conditions is easy. We define $\mathbb{P}_k^n$ by gluing finitely many copies of $\mathbb{A}_k^n$. Since $\mathbb{A}_k^n$ is quasi-compact, reduced, and locally of finite type, so is $\mathbb{P}_k^n$. It remains to check that $\mathbb{P}_k^n$ is irreducible. In fact we have the following topological fact:

Suppose that X has an open cover $\{U_i\}_{i\in I}$ of irreducible spaces such that $U_i \cap U_j \neq \varnothing$ for all $i, j \in I$. Then X is irreducible.

Suppose that X is reducible. There are non-empty open sets V, W such that $V \cap W = \varnothing$. We may assume that $U_i \cap V \neq \varnothing$ and $U_j \cap W \neq \varnothing$. Note that

$$U_i \cap U_j \cap V \cap W = (U_i \cap U_j \cap V) \cap (U_j \cap W) = (U_i \cap V) \cap (U_i \cap U_j) \cap (U_j \cap W)$$

Since U_i is irreducible, we have $(U_i \cap V) \cap (U_i \cap U_j) \neq \varnothing$; since U_j is irreducible, we have $(U_i \cap U_j \cap V) \cap (U_j \cap W) \neq \varnothing$. This contradicts that $V \cap W = \varnothing$. Hence X is irreducible.

Now since each $U_i \cong \mathbb{A}_k^n$ in $\mathbb{P}_k^n$ is irreducible, and $U_i \cap U_j$ is non-empty, we deduce that $\mathbb{P}_k^n$ is irreducible. This finishes the proof that $\mathbb{P}_k^n$ is a variety. Also $\mathbb{P}^n = \overline{U_i}$ so irred.

vi) Let $X \subseteq \mathbb{P}_k^n$ be a projective variety. By definition it is an integral closed subscheme of $\mathbb{P}_k^n$. So it is quasi-compact and locally of finite type. Hence $\mathbb{P}_k^n$ is of finite type over k. We need to prove that X is separated. More generally, we would like to prove that

A closed subscheme X of a separated scheme Y (over any base scheme B) is separated. (quote notes)

By (iii) it suffices to show that $\Delta_{X/B}(X)$ is closed in $X \times_B X$. This follows from that

$$\Delta_{X/B}(X) = \Delta_{Y/B}(Y) \cap (X \times_B X)$$

and that $\Delta_{Y/B}(Y)$ is closed in $Y \times_B Y$.

We conclude that a projective variety is a variety.

For a quasi-projective variety, since it is an irreducible open scheme of a projective variety, by Question 2.(ii), it is also a variety. ☐

Question 4

Fact. $\mathbb{P}^n_k$ *is* ***complete*** *(i.e. proper over k). In this exercise we work over an algebraically closed field k.*

i) In notes, we showed that $\mathbb{A}^1$ is not complete because $\mathbb{A}^1 \times \mathbb{A}^1 \supseteq \mathbb{V}(xy-1) \to \mathbb{A}^1$ fails the **universally closed** condition. Why is this not a problem for $\mathbb{P}^1$ if consider $\mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$?

ii) Let $C \subseteq X$ be a closed subscheme. Prove that X is complete $\implies C$ is complete.

[*Campare in topology: a closed subset of a compact space is compact*]

[*So the fact at the beginning implies also that all projective vorieties are complete.*]

iii) Let $f : X \to Y$ be a morphism of schemes, where X is universally closed and Y is separated (*Hint. graph*). Show that $\operatorname{im} f \subseteq Y$ (*use $f_*\mathcal{O}_X$ on $\operatorname{im} f$ to get scheme*) is closed and universally closed

[*Compare topology: the image of a continuous map from a compact space to a Hausdorff space is closed and compact.*]

iv) Let X be a complete variety. Show that $s \in \Gamma(X, \mathcal{O}_X)$ constant.

[*Hint.* $\Gamma(X, \mathcal{O}_X) = \operatorname{Mor}(X, \mathbb{A}^1)$ *see Sec 2.3 notes.*]

v) Deduce that affine varieties ($\neq$ point, $\varnothing$) are never complete, and that the only global sections of a projective voriety X are constant morphisms $X \to \mathbb{A}^1$.

Proof. Throughout the question, $X \times_k Y$ is short for $X \times_{\operatorname{Spec} k} Y$.

i) The universally closed condition does not fail because $\mathbb{V}(xy-1) \subseteq \mathbb{A}^1_k \times_k \mathbb{A}^1_k$ is closed in $\mathbb{A}^1_k \times_k \mathbb{A}^1_k$ but not in $\mathbb{P}^1_k \times_k \mathbb{A}^1_k$. yes, and it's closure $\mathbb{V}(x_0 y - x_1)$ projects onto $\mathbb{A}^1$ not $\mathbb{A}^1 \setminus 0$

ii) Suppose that X is a complete variety. We need to prove that $C \to \operatorname{Spec} k$ is universally closed. Let $g : Y \to \operatorname{Spec} k$ be any morphism. Since X is universally closed, we know that $f : X \to \operatorname{Spec} k$ is closed, and we have the commutative diagram as below, where $\widetilde{f} : X \times_k Y \to Y$ is closed.

$$\begin{tikzcd} X \times_k Y \arrow[r, "\pi"] \arrow[d, "\widetilde{f}"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & \operatorname{Spec} k \end{tikzcd}$$

Let $i : C \to X$ be the closed immersion. The projection $C \times_k Y \to C$ factors through $X \times_k Y$ via the closed immersion $j : C \times_k Y \to X \times_k Y$ and the projection $\pi : X \times_k Y \to X$. We have the commutative diagram:

$$\begin{tikzcd} C \times_k Y \arrow[r, "j"] \arrow[dr, "\widetilde{f} \circ j"'] & X \times_k Y \arrow[r, "\pi"] \arrow[d, "\widetilde{f}"'] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"] & \operatorname{Spec} k \end{tikzcd}$$

Since $f : X \to \operatorname{Spec} k$ is closed, the induced map $C \to \operatorname{Spec} k$ is also closed. The composite map $\widetilde{f} \circ j$ is closed because both $\widetilde{f}$ and j are closed. The diagram implies that C is universally closed. Hence C is a complete subvariety of X. good

iii) First we prove that $\operatorname{im} f$ is closed. Let $\Gamma_f : X \to X \times_k Y$ be the graph of $f : X \to Y$. $f : X \to Y$ factors as $X \xrightarrow{\Gamma_f} X \times_k Y \xrightarrow{\pi} Y$. Since Y is separated, a claim from the notes shows that Γ_f is a closed immersion. In particular $\Gamma_f(X) \subseteq X \times_k Y$ is closed. Since X is universally closed, $\pi : X \times_k Y \to Y$ is closed. Then $\operatorname{im} f = \pi \circ \Gamma_f(X)$ is closed.

Next we prove that $\operatorname{im} f$ is universally closed over k (*for this part I think the separatedness of Y is* *unnecessary*). Let $\alpha : X \to \operatorname{Spec} k$ and $\beta : \operatorname{im} f \to \operatorname{Spec} k$ be the morphisms. Let $g : Z \to \operatorname{Spec} k$ be any morphism. We look at the commutative diagram of base changes:

$$\begin{tikzcd}
X \times_k Z \arrow[r, "\widetilde{\alpha}"'] \arrow[d, "\pi_X"'] \arrow[rr, bend left, "\widetilde{f}"] & Z \arrow[d, "g"] & \operatorname{im} f \times_k Z \arrow[l, "\widetilde{\beta}"'] \arrow[d, "\pi_Y"] \\
X \arrow[r, "\alpha"] \arrow[rr, bend right, "f"'] & \operatorname{Spec} k & \operatorname{im} f \arrow[l, "\beta"']
\end{tikzcd}$$

$f : X \to \operatorname{im} f$ is surjective, then so is $\widetilde{f} : X \times_k Z \to \operatorname{im} f \times_k Z$. Let $C \subseteq \operatorname{im} f \times_k Z$ be a closed subset. Then $\widetilde{\beta}(C) = \widetilde{\alpha}(\widetilde{f}^{-1}(C))$ is closed because $\widetilde{f}$ is surjective and continuous, and α is closed. Hence $\widetilde{\beta}$ is a closed map. We deduce that $\operatorname{im} f$ is universally closed over k. good

iv) We know that $\mathbb{A}^1_k = \operatorname{Spec} k[x]$. From Example 1 of Section 2.3 of the notes, we have a bijection

as k-algs

$$\operatorname{Mor}(X, \mathbb{A}^1_k) \longleftrightarrow \operatorname{Hom}_k(k[x], \mathcal{O}_X(X)) \cong \mathcal{O}_X(X)$$

Spec

Since X is complete, it is universally closed. We know that $\mathbb{A}^1_k$ is separated. Then for any morphism $f : X \to \mathbb{A}^1_k$, by (iii) $\operatorname{im} f$ is closed and universally closed in $\mathbb{A}^1_k$. Since X is irreducible, so is $\operatorname{im} f$. Then we find that $\operatorname{im} f = \mathbb{V}(x - a)$ for some $a \in k$ (this is a singleton on the affine line). Hence $\Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X) \cong k$. The global sections are constant morphisms on X.

v) Suppose that $Y \subseteq \mathbb{A}^n_k$ is an affine variety with $\operatorname{card}(Y) > 1$. We take two distinct closed points $\boldsymbol{a} = \mathbb{V}(\langle x_1 - a_1, ..., x_n - a_n \rangle)$ and $\boldsymbol{b} = \mathbb{V}(\langle x_1 - b_1, ..., x_n - b_n \rangle)$ in Y. We may assume that $a_i \neq b_i$ for some i. Then $x_i \in \mathcal{O}_Y(Y)$ is a non-constant global section. By (iv), Y is not complete.

Suppose that X is a projective variety. We claim that X is complete. Since X is an integral closed subscheme of some $\mathbb{P}^n_k$, by (ii) it suffices to prove that $\mathbb{P}^n_k$ is complete. (*I don't know if this proof is examinable. It is not shown in the notes. In* Hartshorne *this follows from Theorem II.4.9, which is a corollary of the* ***valuation criterion of properness****. So I choose not to go into details here...*) Now by (iv), we know that the global sections of X are constant morphisms. $\square$

just quote it, it's given as a fact.

Question 5

Note that any "commutative diagram" in a category C *can be thought of as a functor* $F : \mathsf{I} \to \mathsf{C}$ *where the objects of* I *are the positions* i *in the diagram (where you place some object* $F(i) = C_i \in \mathsf{C}$*), the morphisms of* I *are the arrows of the diagram (together with all identity morphs* $i \to i$ *and composites)*

i) What is the functor of points interpretation of $\varprojlim$, $\varinjlim$? (*Hint. for* $\varinjlim$ *consider* I^{op} *and* h^* *not* h_*)

ii) Explain briefly why the product, fibre product, gluing of sheaves are limits, and the coproduct, pushout, gluing of schemes are colimits (*e.g. every scheme* $= \varinjlim$ *of its affine opens*)

iii) Suppose f, g are adjoint functors. Show that left adjoints commute with colimits, right adjoints commute with limits.

fair.