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Problem Sheet 3

CFT in two dimensions

Conformal Field Theory

1. A
2. A-
3. B
4. A
5. B/C
6. NA

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Remark. In a 2D conformal field theory, the global conformal transformations on $\mathbb{CP}^1$ are exactly the Möbius transformations, which form the group $\mathrm{PSL}(2, \mathbb{C})$.

Question 1

(a) Given four point in the complex plane $z_1, \cdots z_4$ show that the cross-ratio η defined in the lectures is invariant under global conformal transformations.

(b) Find a global transformation that maps the points $(0, \mathrm{i}, 2)$ to the points $(0, 1, \infty)$.

Proof. (a) The cross-ratio is defined by

$$\eta := \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}.$$

It is clear that η is invariant under translations, dilations, and rigid rotations. Indeed, such global transformations can be represented by $f(z) = az + b$ for some $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$. We have

$$f_*\eta = \frac{(f(z_1) - f(z_2))(f(z_3) - f(z_4))}{(f(z_1) - f(z_3))(f(z_2) - f(z_4))} = \frac{a^2(z_1 - z_2)(z_3 - z_4)}{a^2(z_1 - z_3)(z_2 - z_4)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} = \eta.$$

Furthermore, η is invariant under inversion $I(z) = 1/z$:

$$I_*\eta = \frac{(1/z_1 - 1/z_2)(1/z_3 - 1/z_4)}{(1/z_1 - 1/z_3)(1/z_2 - 1/z_4)} = \frac{(z_2 - z_1)(z_4 - z_3)}{(z_3 - z_1)(z_4 - z_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} = \eta.$$

Therefore η is invariant under any global conformal transformations (where we used the fact that a special conformal transformation is a composition of some translations and the inversion). OK

(b) Consider a Möbius transformation $T(z) = \dfrac{az + b}{cz + d}$ $(ad - bc \neq 0)$, which is conformal on the Riemann sphere $\mathbb{CP}^1$. We want $T(0) = 0$, $T(\mathrm{i}) = 1$ and $T(2) = \infty$. This implies that $b/d = 0$, $2c + d = 0$, and $a\mathrm{i} + b = c\mathrm{i} + d$. Solving these equations we obtain $a = -\left(\frac{1}{2} + \mathrm{i}\right) d$, $b = 0$, $c = -\frac{1}{2}d$, and $d \neq 0$. Therefore the Möbius transformation is given by

$$T(z) = (1 + 2\mathrm{i})\frac{z}{z - 2}.$$

□

Question 2

(vertex operator)

Consider a free scalar field in two dimensions $\varphi(x)$ and the operator $\mathcal{O}_\alpha = \,:e^{\mathrm{i}\alpha\varphi(z)}:$, where α is a real constant. Focusing only in its holomorphic dependence, compute the OPE of this operator with the stress tensor and verify that it is a primary operator of a given weight that you should compute.

[*Note: The normal ordering symbol is meant to remind us not to Wick contract two scalar fields within the operator.*]

Show that the two point function of such operators behaves as it should.

Proof. For the free scalar field CFT, the (holomorphic[1] part of) propagator is given by

$$\langle \varphi(z)\varphi(w) \rangle = -\frac{1}{4\pi} \ln(z - w),$$

[1] I suppose this should be called *meromorphic* in the context of complex analysis, but *holomorphic* is fine in the context of Riemann surfaces...

and the stress-energy tensor is given by

$$T(z) = -2\pi :\partial\varphi(z)\partial\varphi(z): =: -2\pi \lim_{w\to z} \left(\partial\varphi(z)\partial\varphi(w) - \langle\partial\varphi(z)\partial\varphi(w)\rangle\right)$$
$$= -2\pi \lim_{w\to z} \left(\partial\varphi(z)\partial\varphi(w) + \frac{1}{4\pi}\frac{1}{(z-w)^2}\right).$$

By performing Wick contractions, the OPE $T(z):e^{i\alpha\varphi(w)}:$ is given by:

$$T(z)\mathcal{O}_\alpha(w) = -2\pi :\partial\varphi(z)\partial\varphi(z)::e^{i\alpha\varphi(w)}:$$
$$= -2\pi \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} :\partial\varphi(z)\partial\varphi(z)::\varphi(w)^n:$$
$$= -2\pi \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \left(:\partial\varphi(z)\varphi(w)^n: + 2n\langle\partial\varphi(z)\varphi(w)\rangle :\partial\varphi(z)\varphi(w)^{n-1}: \right.$$
$$\left. + n(n-1)\langle\partial\varphi(z)\varphi(w)\rangle\langle\partial\varphi(z)\varphi(w)\rangle :\varphi(w)^{n-2}:\right)$$
$$= -2\pi \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \left(:\partial\varphi(z)\varphi(w)^n: + \left(-\frac{2n}{4\pi}\frac{1}{z-w}\right) :\partial\varphi(z)\varphi(w)^{n-1}: + \frac{n(n-1)}{(4\pi)^2}\frac{1}{(z-w)^2} :\varphi(w)^{n-2}:\right)$$
$$= \frac{\alpha^2}{8\pi}\frac{:e^{i\alpha\varphi(w)}:}{(z-w)^2} + i\alpha\frac{:\partial\varphi(z)\, e^{i\alpha\varphi(w)}:}{z-w} + \text{entire functions.}$$
$$= \frac{\alpha^2}{8\pi}\frac{:e^{i\alpha\varphi(w)}:}{(z-w)^2} + \frac{\partial_w :e^{i\alpha\varphi(w)}:}{z-w} + \text{entire functions.}$$

$\partial_z\varphi = \partial_w\varphi + (z-w)\,\partial^2\varphi$ ← regular, trick for single pole

Good

Hence $\mathcal{O}_\alpha = :e^{i\alpha\varphi}:$ is a primary operator with conformal weight $h = \dfrac{\alpha^2}{8\pi}$. The OPE $\mathcal{O}_\alpha(z)\mathcal{O}_\alpha(w)$ is given by performing infinitely many times of Wick contractions:

$$\mathcal{O}_\alpha(z)\mathcal{O}_\alpha(w) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(i\alpha)^{n+m}}{n!m!} :\varphi(z)^n::\varphi(w)^m:$$
$$= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty} \frac{(i\alpha)^{p+q+2r}}{(p+r)!(q+r)!}\frac{(p+r)!}{p!}\frac{(q+r)!}{q!}\frac{1}{r!}\langle\varphi(z)\varphi(w)\rangle^r :\varphi(z)^p\varphi(w)^q:$$
$$= \left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(i\alpha)^{p+q}}{p!q!} :\varphi(z)^p\varphi(q)^q:\right)\left(\sum_{r=0}^{\infty}\frac{(-\alpha^2)^r}{r!}\langle\varphi(z)\varphi(w)\rangle^r\right)$$
$$= e^{-\alpha^2\langle\varphi(z)\varphi(w)\rangle} :e^{i\alpha\varphi(z)}\, e^{i\alpha\varphi(w)}:$$
$$= (z-w)^{\frac{\alpha^2}{4\pi}} :e^{i\alpha\varphi(z)}\, e^{i\alpha\varphi(w)}:$$
$$= (z-w)^{\frac{\alpha^2}{4\pi}} :\mathcal{O}_\alpha(z)\mathcal{O}_\alpha(w):.$$

OPE $\mathcal{O}_\alpha\mathcal{O}_\alpha = \langle\mathcal{O}_\alpha\mathcal{O}_\alpha\rangle$
expectation is taken by default...?

Therefore the 2-point correlation function is given by

What is the 1 doing here?

$$\langle\mathcal{O}_\alpha(z)\mathcal{O}_\alpha(w)\rangle = \mathcal{O}_\alpha(z)\mathcal{O}_\alpha(w) - :\mathcal{O}_\alpha(z)\mathcal{O}_\alpha(w): = \left(1-(z-w)^{\frac{\alpha^2}{4\pi}}\right) :\mathcal{O}_\alpha(z)\mathcal{O}_\alpha(w):.$$

The result does not look right...Perhaps we need another 2-point correlation function:

$$\langle\mathcal{O}_\alpha(z)\mathcal{O}_{-\alpha}(w)\rangle = (z-w)^{-\alpha^2/4\pi} :\mathcal{O}_\alpha(z)\mathcal{O}_{-\alpha}(w): + \text{entire functions.}$$ □

That's right, the 2 pt function is between the operator and its conjugate

State operator correspondence
$\mathcal{O}_\alpha(0)|0\rangle = |h_\alpha\rangle \Rightarrow \langle h_\alpha| = \lim_{z\to\infty} z^{2h_\alpha}\langle 0|\mathcal{O}_\alpha^\dagger(z)$

String interaction n-pt function.
$\langle\mathcal{O}_{\alpha_1}\cdots\mathcal{O}_{\alpha_n}\rangle$ $\quad \sum_i \alpha_i = 0$
Momentum conservation!

Question 3

(a) Calculate the four-point function $\langle \partial\varphi\partial\varphi\partial\varphi\partial\varphi \rangle$ for the free two-dimensional boson, using Wick contraction. Compare it with the general expression given in the lectures and determine the function $g(\eta)$ in this case.

(b) Calculate now the correlator $\langle T(z)\partial\varphi\partial\varphi\partial\varphi\partial\varphi \rangle$, where $T(z)$ is the holomorphic stress tensor given in the lectures, using Wick contraction. Verify the conformal Ward identities for this case.

Proof. (a) We know that

$$\langle \partial\varphi(z)\partial\varphi(w) \rangle = -\frac{1}{4\pi}\frac{1}{(z-w)^2}.$$

By Wick contraction, the 4-point function is given by

$$\begin{aligned}\langle \partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3)\partial\varphi(z_4) \rangle &= \frac{1}{4}\sum_{\sigma\in S_4} \langle \partial\varphi(z_{\sigma(1)})\partial\varphi(z_{\sigma(2)}) \rangle : \partial\varphi(z_{\sigma(3)})\partial\varphi(z_{\sigma(4)}) : \\ &+ \frac{1}{8}\sum_{\sigma\in S_4} \langle \partial\varphi(z_{\sigma(1)})\partial\varphi(z_{\sigma(2)}) \rangle \langle \partial\varphi(z_{\sigma(3)})\partial\varphi(z_{\sigma(4)}) \rangle \\ &= \frac{1}{8}\sum_{\sigma\in S_4} \frac{1}{(4\pi)^2}\frac{1}{(z_{\sigma(1)}-z_{\sigma(2)})^2(z_{\sigma(3)}-z_{\sigma(4)})^2} - \frac{1}{4}\sum_{\sigma\in S_4}\frac{1}{4\pi}\frac{:\partial\varphi(z_{\sigma(3)})\partial\varphi(z_{\sigma(4)}):}{(z_{\sigma(1)}-z_{\sigma(2)})^2}.\end{aligned}$$

The most singular term in the expression is given by:

I don't see what the term with the : … : is doing there!

$$\begin{aligned}&\frac{1}{(4\pi)^2}\left(\frac{1}{z_{12}^2 z_{34}^2}+\frac{1}{z_{13}^2 z_{24}^2}+\frac{1}{z_{14}^2 z_{23}^2}\right)\\ &= \frac{1}{(4\pi)^2}\frac{1}{(z_{12}z_{13}z_{14}z_{23}z_{24}z_{34})^{2/3}}\left(\left(\frac{z_{13}z_{24}}{z_{12}z_{34}}\right)^{2/3}\left(\frac{z_{14}z_{23}}{z_{12}z_{34}}\right)^{2/3}+\left(\frac{z_{12}z_{34}}{z_{13}z_{24}}\right)^{2/3}\left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{2/3}+\left(\frac{z_{12}z_{34}}{z_{14}z_{23}}\right)^{2/3}\left(\frac{z_{13}z_{24}}{z_{14}z_{23}}\right)^{2/3}\right)\\ &= \frac{1}{(4\pi)^2}\frac{1}{(z_{12}z_{13}z_{14}z_{23}z_{24}z_{34})^{2/3}}\left(\alpha^{2/3}\beta^{2/3}+\alpha^{-2/3}\gamma^{2/3}+\beta^{-2/3}\gamma^{-2/3}\right)\end{aligned}$$

where $\alpha := \frac{z_{13}z_{24}}{z_{12}z_{34}}$, $\beta := \frac{z_{14}z_{23}}{z_{12}z_{34}}$, $\gamma := \frac{z_{14}z_{23}}{z_{13}z_{24}}$, and $z_{ij} := z_i - z_j$ for $i,j = 1,...,4$. Note that α, β and γ are cross-ratios and are conformally invariant. We may apply a Möbius transformation such that $(z_1,z_2,z_3,z_4) \mapsto (0,1,\eta,\infty)$. Then $\alpha = \eta$, $\beta = \eta - 1$ and $\gamma = \frac{\eta-1}{\eta}$. Therefore

$$\alpha^{2/3}\beta^{2/3}+\alpha^{-2/3}\gamma^{2/3}+\beta^{-2/3}\gamma^{-2/3} = \eta^{4/3}(\eta-1)^{2/3}+\frac{(\eta-1)^{2/3}}{\eta^{4/3}}+\frac{\eta^{2/3}}{(\eta-1)^{4/3}} = .$$

In summary, the 4-point function $\langle \partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3)\partial\varphi(z_4) \rangle$ is given by

$$\frac{1}{(4\pi)^2}\left(\eta^{2/3}(\eta-1)^{2/3}+\frac{(\eta-1)^{2/3}}{\eta^{4/3}}+\frac{\eta^{2/3}}{(\eta-1)^{4/3}}\right)\frac{1}{(z_{12}z_{13}z_{14}z_{23}z_{24}z_{34})^{2/3}}+\cdots.$$

By comparing this expression with the general expression in the notes:

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\phi_4(z_4,\bar{z}_4) \rangle = g(\eta,\bar{\eta})\prod_{i<j} z_{ij}^{h/3-h_i-h_j}\bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j}$$

We obtain that

$$g(\eta) = \frac{1}{(4\pi)^2}\left(\eta^{2/3}(\eta-1)^{2/3}+\frac{(\eta-1)^{2/3}}{\eta^{4/3}}+\frac{\eta^{2/3}}{(\eta-1)^{4/3}}\right).$$

Good!

Note that we have neglected the anti-holomorphic part in the above calculation. The terms are symmetric to the holomorphic ones.

(b)

$$T'(z') = \left(\frac{\partial f}{\partial z}\right)^{-2}\left(T(z) - \frac{c}{12}\{f(z), z\}\right)$$

□

Question 4

Show that the Schwarzian derivative vanishes when restricted to global conformal transformations.

Proof. The Scharzian derivative is given by

$$\{f(z), z\} = Sf(z) := \frac{1}{f'(z)^2}\left(f'(z)f'''(z) - \frac{3}{2}f''(z)^2\right).$$

Let $f(z) = \dfrac{az+b}{cz+d}$ $(ad - bc \neq 0)$ be a Möbius transformation. Then:

$$f'(z) = \frac{ad-bc}{(cz+d)^2}, \qquad f''(z) = -\frac{2c(ad-bc)}{(cz+d)^3} = \frac{-2cf'(z)}{cz+d}, \qquad f'''(z) = \frac{6c^2(ad-bc)}{(cz+d)^4} = \frac{6c^2 f'(z)}{(cz+d)^2}.$$

Therefore the Schwarzian of f is given by

$$Sf(z) = \frac{1}{f'(z)^2}\left(f'(z)\frac{6c^2 f'(z)}{(cz+d)^2} - \frac{3}{2}\left(\frac{-2cf'(z)}{cz+d}\right)^2\right) = 0.$$

Good!

□

Question 5

Given a Virasoro primary $|h\rangle$ such that

$$L_0|h\rangle = h|h\rangle, \quad \langle h \mid h\rangle = 1$$

Compute the inner products between all level two descendants and their conjugates.

Proof. The Level 2 descendants are $L_{-2}|h\rangle$ and $L_{-1}^2|h\rangle$. We can compute their norms with the known Lie brackets:

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}.$$

For $L_{-2}|h\rangle$, we have:

$$\|L_{-2}|h\rangle\|^2 = \langle h| L_2 L_{-2}|h\rangle = \langle h| [L_2, L_{-2}]|h\rangle = \langle h|\left(4L_0 + \frac{c}{2}\right)|h\rangle = 4h + \frac{c}{2}.$$

For $L_{-1}^2|h\rangle$, we have:

$$\left\|L_{-1}^2|h\rangle\right\|^2 = \langle h| L_1^2 L_{-1}^2|h\rangle = 2\langle h| [L_1, L_{-1}]^2|h\rangle = 2\langle h| (2L_0)^2|h\rangle = 8h^2.$$

□

This equality is wrong

should be $8h^2+4h$

& $\langle h| L_2 L_{-1}^2 |h\rangle = 6h$

What about the mixed term?

Question 6. Dentity Virasoro conformal block

Consider two identical operators of conformal weight $(h, \bar{h})$ such that they are canonically normalized

$$\langle \phi_{h,h}(z,\bar{z})\phi_{h,h}(0)\rangle = \frac{1}{z^{2h}\bar{z}^{2h}}$$

Consider the OPE (6.25) in the lecture notes, and focus in the identity operator plus its Virasoro descendants.

(a) Compute the OPE coefficients $C_{12}^{Id,(k,\bar{k})}$ up to level two.

(b) Use the result of part (a) to compute the small z expansion of the Virasoro conformal block for the identity operator.

Proof. (a) The identity operator id: $z \mapsto (|\psi\rangle \mapsto |\psi\rangle)$ has conformal weight $(h, \bar{h}) = (0, 0)$. □

Neglect anti-holomorphic part.

$$\varphi(z)\varphi(0) = \sum_{k=-1}^{-\infty} z^{-2h+|k|}\, id^{(k)}(0)\, C_{\varphi\varphi}{}^{id,\{k\}}$$

$$= \langle id(w)\varphi(z)\varphi(0)\rangle = \frac{1}{z^{2h}}$$

Descendents of id : $L_{-1} id = 0$, $L_{-2} id = T$

$$\langle T(w)\varphi(z)\varphi(0)\rangle \overset{\text{Ward}}{=\!=} \frac{h}{w^2(w-z)^2 z^{2h-2}}$$

$$\simeq \underbrace{\frac{h}{w^4 z^{2h-2}}}_{\text{OPE with } T} + \underbrace{\cdots}_{\text{OPE with descendents of } T}$$

$$\langle T(w)\cdot C_{\varphi\varphi}{}^{id,\{2\}} z^{-2h+2} T(0)\rangle = C_{\varphi\varphi}{}^{id,\{2\}} z^{2-2h} \left\langle \frac{c/2}{w^4} + \frac{2T(0)}{w^2} + \frac{\partial T(0)}{w} \right\rangle$$

$$= C_{\varphi\varphi}{}^{id,\{2\}} \frac{c}{2} \cdot \frac{1}{w^4 z^{2h-2}}$$

$$\Rightarrow C_{\varphi\varphi}{}^{id,\{2\}} = \frac{2h}{c}$$

(b) $$\mathcal{F}_h(p|z) = z^{h_p - 2h} \sum_{\{k\}} \beta_p^{\{k\}} z^k \frac{\langle h|\varphi(1) L_{k_1}\cdots L_{k_m}|h_p\rangle}{\langle h|\varphi(1)|h_p\rangle}$$

$p = 0$, $h_p = 0$, $|h_p\rangle = |0\rangle$

$$|h\rangle = \lim_{z\to 0}\varphi(z)|0\rangle = \lim_{z\to\infty} z^{2h}\langle 0|\varphi(z)$$

$$k=0: \quad z^{-2h}\cdot z^0 \frac{\langle h|\varphi(1)|0\rangle}{\langle h|\varphi(1)|0\rangle} = z^{-2h}$$

$$k=1: \quad 0$$

$$k=2: \quad z^{0-2h}\underbrace{\beta_0^{\{2\}}}_{=2h/c} z^2 \frac{\langle h|\varphi(1)L_{-2}|0\rangle}{\langle h|\varphi(1)|0\rangle}$$

$$\to \langle h|[\varphi(1), L_{-2}]|0\rangle = \lim_{w\to 1}\langle h|(-h(-2+1)w^{-2}\varphi(w) + w^{-2+1}\partial\varphi(w)|0\rangle$$

$$= \lim_{w\to 1}\lim_{z\to\infty} z^{2h}\left(\frac{h}{w^2}\cdot\frac{1}{(z-w)^{2h}} - \frac{1}{w}\frac{\partial}{\partial w}\frac{1}{(z-w)^{2h}}\right)|0\rangle$$

$$= h$$

$$\hookrightarrow \lim_{y\to\infty} y^{2h}\langle 0|\varphi(y)\varphi(1)|0\rangle$$

$$= \lim_{y\to\infty} y^{2h}\cdot y^{-2h} = 1$$

$$\downarrow$$

$$= z^{-2h+2}\frac{2h^2}{c}$$

Global Virasoro algebra.

1) $L_n = z^{n+1} \partial_z$.

- $z = 0$: L_n regular for $n \geq -1$
- $z = \infty$: L_n regular for $n \leq 1$

$\Rightarrow$ Only global generators are $L_{\pm 1}$, L_0

($L_{-1} = P$, $L_0 = D$, $L_1 = K$, sl_2-rep.)

2) $z(t)$: $\frac{dz}{dt} = z^{n+1} \Rightarrow z(t) = (-n(t+c))^{-1/n}$

No branch points for $n = 0, \pm 1$

3) $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ holomorphic

Riemann-Roch $f = \frac{p(z)}{q(z)} \in \mathbb{C}(z)$