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Problem Sheet 1

String Theory II

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A-

Question 1. Super-Virasoro Algebra

Consider the algebra of oscillators for the RNS-string

$$[\alpha_n^\mu, \alpha_m^\nu] = n\eta^{\mu\nu}\delta_{n+m,0}, \quad n, m \in \mathbb{Z}$$
$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0}, \quad r, s \in \mathbb{Z}+\phi$$

where $\phi = 0, \frac{1}{2}$ for the R, NS-sector. Define the generators of the Super-Virasoro algebra by the normal ordered expressions

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} :\alpha_{-m}\cdot\alpha_{m+n}: + \sum_{r\in\mathbb{Z}+\phi}(r+n/2):b_{-r}b_{n+r}:$$
$$G_r = \sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot b_{r+m},$$

for $n \in \mathbb{Z}$ and $r \in \mathbb{Z}+\phi$. Determine the algebra that these operators generate, i.e. compute $[L_m, L_n], [L_m, G_r]$ and $\{G_r, G_r\}$.

Proof. With slightly abuse of notation, the index-free super-brackets are given by

$$[\alpha_n, \alpha_m] := \alpha_n\cdot\alpha_m - \alpha_m\cdot\alpha_n = nd\delta_{m+n,0}, \qquad \{b_r, b_s\} := b_r\cdot b_s + b_s\cdot b_r = d\delta_{r+s,0},$$

where $d = \delta^\mu{}_\mu$ is the spacetime dimension. Let $L_m = L_m^{(\alpha)} + L_m^{(b)}$, where

$$L_m^{(\alpha)} = \frac{1}{2}\sum_{k\in\mathbb{Z}} :\alpha_{-k}\cdot\alpha_{m+k}:, \qquad L_m^{(b)} = \frac{1}{2}\sum_{k\in\mathbb{Z}+\phi}\left(k+\frac{m}{2}\right):b_{-k}\cdot b_{m+k}:.$$

We know from String Theory I (Sheet 2 Question 2) that $L_m^{(\alpha)}$ satisfies the ordinary Virasoro algebra:

$$[L_m^{(\alpha)}, L_n^{(\alpha)}] = (m-n)L_{m+n}^{(\alpha)} + \frac{d}{12}(m^3-m)\delta_{m+n,0},$$

We perform a similar computation for $L_m^{(b)}$. Note that the commutator:

$$\begin{aligned}
[b_{-k}b_{m+k}, b_{-\ell}b_{n+\ell}] &= b_{-k}\{b_{m+k}, b_{-\ell}\}b_{n+\ell} + b_{-\ell}b_{-k}\{b_{m+k}, b_{n+\ell}\} - \{b_{-k}, b_{-\ell}\}b_{m+k}b_{n+\ell} - b_{-\ell}\{b_{-k}, b_{n+\ell}\}b_{m+k} \\
&= (b_{-k}b_{k+m+n}\delta_{m+k-\ell,0} + b_{k+m+n}b_{-k}\delta_{m+n+k+\ell,0} - b_{m+k}b_{n-k}\delta_{k+\ell,0} - b_{n-k}b_{m+k}\delta_{n+\ell-k,0})
\end{aligned}$$

For $m+n \neq 0$, we can drop the normal ordering and compute the commutator:

$$\begin{aligned}
\left[L_m^{(b)}, L_n^{(b)}\right] &= \frac{1}{4}\sum_{k\in\mathbb{Z}+\phi}\sum_{\ell\in\mathbb{Z}+\phi}\left(k+\frac{m}{2}\right)\left(l+\frac{n}{2}\right)[b_{-k}b_{m+k}, b_{-\ell}b_{n+\ell}] \\
&= \frac{1}{4}\sum_{k\in\mathbb{Z}+\phi}\left(k+\frac{m}{2}\right)\left(\left(m+k+\frac{n}{2}\right)b_{-k}b_{k+m+n} + \left(-m-k-\frac{n}{2}\right)b_{k+m+n}b_{-k}\right. \\
&\qquad\qquad \left. - \left(-k+\frac{n}{2}\right)b_{m+k}b_{n-k} - \left(k-\frac{n}{2}\right)b_{n-k}b_{m+k}\right) \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}+\phi}\left(k+\frac{m}{2}\right)\left(\left(m+k+\frac{n}{2}\right)b_{-k}b_{m+n+k} - \left(k-\frac{n}{2}\right)b_{n-k}b_{m+k}\right) \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}+\phi}\left(\left(k+\frac{m}{2}\right)\left(m+k+\frac{n}{2}\right) - \left(k+\frac{n}{2}\right)\left(k+\frac{m}{2}+n\right)\right)b_{-k}b_{m+n+k} \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}+\phi}\left(k+\frac{m+n}{2}\right)(m-n)b_{-k}b_{m+n+k}
\end{aligned}$$

$$= (m-n)L^{(b)}_{m+n}$$

Therefore we have $[L^{(b)}_m, L^{(b)}_n] = (m-n)L^{(b)}_{m+n} + \beta(m)\delta_{m+n,0}$, where $\beta(m) \in \mathbb{C}$ is a constant due to normal ordering.

The computation for $[L_m, G_r]$:

$$\begin{aligned}
\left[L^{(\alpha)}_m, G_r\right] &= \frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}} [:\alpha_{-k}\alpha_{m+k}:, \alpha_{-\ell}b_{r+\ell}] \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}} [\alpha_{-k}\alpha_{m+k}, \alpha_{-\ell}b_{r+\ell}] \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}} \left([\alpha_{-k}, \alpha_{-\ell}]\,\alpha_{m+k}b_{r+l} + \alpha_{-k}\,[\alpha_{m+k}, \alpha_{-\ell}]\,b_{r+\ell}\right) \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}} \left(-k\alpha_{m+k}b_{r+\ell}\delta_{k+\ell,0} + (m+k)\alpha_{-k}b_{r+\ell}\delta_{m+k-\ell,0}\right) \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}} \left(-k\alpha_{m+k}b_{r-k} + (m+k)\alpha_{-k}b_{r+m+k}\right) \\
&= \sum_{k\in\mathbb{Z}}(m+k)\alpha_{-k}b_{r+m+k} \\
\left[L^{(b)}_m, G_r\right] &= \frac{1}{2}\sum_{k\in\mathbb{Z}+\phi}\sum_{\ell\in\mathbb{Z}} \left(k+\frac{m}{2}\right)[:b_{-k}b_{m+k}:, \alpha_{-\ell}b_{r+\ell}] \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}+\phi}\sum_{\ell\in\mathbb{Z}} \left(k+\frac{m}{2}\right)[b_{-k}b_{m+k}, \alpha_{-\ell}b_{r+\ell}] \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}+\phi}\sum_{\ell\in\mathbb{Z}} \left(k+\frac{m}{2}\right)\left(\alpha_{-l}b_{-k}\,\{b_{m+k}, b_{r+l}\} + \alpha_{-\ell}\{b_{-k}, b_{r+\ell}\}b_{m+k}\right) \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}+\phi}\sum_{\ell\in\mathbb{Z}} \left(k+\frac{m}{2}\right)\left(\alpha_{-\ell}b_{-k}\delta_{m+k+r+l,0} + \alpha_{-\ell}b_{m+k}\delta_{r+\ell-k,0}\right) \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}+\phi} \left(k+\frac{m}{2}\right)\left(\alpha_{m+r+k}b_{-k} + \alpha_{r-k}b_{m+k}\right) \\
&= -\sum_{k\in\mathbb{Z}} \left(k+\frac{m}{2}+r\right)\alpha_{-k}b_{r+m+k}
\end{aligned}$$

Therefore we have $[L_m, G_r] = \sum_{k\in\mathbb{Z}}\left(\frac{m}{2}-r\right)\alpha_{-k}b_{r+m+k} = \left(\frac{m}{2}-r\right)G_{m+r}$.

The computation of $\{G_r, G_s\}$:

$$\begin{aligned}
\{G_r, G_s\} &= \sum_{k\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}} \{\alpha_{-k}b_{r+k}, \alpha_{-\ell}b_{s+\ell}\} \\
&= \sum_{k\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}} \left([\alpha^\mu_{-k}, \alpha^\rho_{-\ell}]b^\nu_{r+k}b^\sigma_{s+\ell} + \alpha^\rho_{-\ell}\alpha^\mu_{-k}\,\{b^\nu_{r+k}, b^\sigma_{s+\ell}\}\right)\eta_{\mu\nu}\eta_{\rho\sigma} \\
&= \sum_{\ell\in\mathbb{Z}}\alpha_{-\ell}\cdot\alpha_{r+s+\ell} + \sum_{\ell\in\mathbb{Z}+\phi}(\ell+r)b_{-\ell}\cdot b_{r+s+\ell} \\
&= \sum_{\ell\in\mathbb{Z}}\alpha_{-\ell}\cdot\alpha_{r+s+\ell} + \sum_{\ell\in\mathbb{Z}+\phi}\left(\ell+\frac{r+s}{2}\right)b_{-\ell}\cdot b_{r+s+\ell} \\
&= 2L_{r+s} + g(r)\delta_{r+s,0},
\end{aligned}$$

where $g(r) \in \mathbb{C}$ is a constant due to the normal ordering.

Finally, the method of finding $\beta(m)$ and $g(r)$ is similar to that for the ordinary Virasoro algebra, which is implemented in Question 2 of Sheet 2 of String Theory I.

For $\beta(m)$, we note that it is an odd function. The graded Jacobi identity for $L_m^{(b)}$ implies that $\beta(m)$ takes the form $\beta_1 m + \beta_3 m^3$ for some $\beta_1, \beta_3 \in \mathbb{C}$.

For $g(r)$, we note that it is an even function. The graded Jacobi identity for G_r implies that $g(r)$ takes the form $g_0 + g_2 r^2$ for some $g_0, g_2 \in \mathbb{C}$.

With some more computations (evaluating the expectation of graded brackets in the ground state, I can't do this calculation due to time constraint...) we can find the super-Virasoro algebra:

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{d}{8}(m^3 - 2\phi m)\delta_{m+n,0}, \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r}, \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{d}{2}\left(r^2 - \frac{\phi}{2}\right)\delta_{r+s,0}.
\end{aligned}
$$

√

□

B

Question 2. Supersymmetry of the RNS-String

Consider the superconformal gauge-fixed RNS-string action

$$
S = S_B + S_F = \frac{1}{2\pi}\int \mathrm{d}^2\sigma \left(\frac{2}{\alpha'}\partial_+ X \cdot \partial_- X + \mathrm{i}\left(\psi_+ \cdot \partial_- \psi_+ + \psi_- \cdot \partial_+ \psi_-\right)\right)
$$

Show that this action is invariant under the global supersymmetry transformations

$$
\begin{aligned}
\sqrt{\frac{2}{\alpha'}}\delta_\epsilon X^\mu &= \mathrm{i}\bar{\epsilon}\psi^\mu = \mathrm{i}\left(\epsilon^+\psi_+^\mu + \epsilon^-\psi_-^\mu\right) \\
\delta_\epsilon \psi_+^\mu &= -\sqrt{\frac{2}{\alpha'}}\epsilon^+\partial_+ X^\mu \\
\delta_\epsilon \psi_-^\mu &= -\sqrt{\frac{2}{\alpha'}}\epsilon^-\partial_- X^\mu
\end{aligned}
$$

Proof. The gauge-fixed action has the following constraints:

$$
\partial_+\partial_- X^\mu = 0; \qquad \partial_-\psi_+^\mu = \partial_+\psi_-^\mu = 0, \qquad \partial_-\epsilon^+ = \partial_+\epsilon^- = 0.
$$

Therefore we have

$$
\begin{aligned}
\delta_\epsilon\left(\frac{2}{\alpha'}\partial_+ X \cdot \partial_- X\right) &= \mathrm{i}\sqrt{\frac{2}{\alpha'}}\left(\left(\partial_+(\epsilon^+\psi_+) + \partial_+(\epsilon^-\psi_-)\right)\cdot\partial_- X + \left(\partial_-(\epsilon^+\psi_+) + \partial_-(\epsilon^-\psi_-)\right)\cdot\partial_+ X\right) \\
\delta_\epsilon\left(\mathrm{i}\psi_+ \cdot \partial_-\psi_+\right) &= -\mathrm{i}\sqrt{\frac{2}{\alpha'}}\left(\epsilon^+\partial_+ X \cdot \partial_-\psi_+ + \psi_+ \cdot \partial_-(\epsilon^+\partial_+ X)\right) \\
\delta_\epsilon\left(\mathrm{i}\psi_- \cdot \partial_+\psi_-\right) &= -\mathrm{i}\sqrt{\frac{2}{\alpha'}}\left(\epsilon^-\partial_- X \cdot \partial_+\psi_- + \psi_- \cdot \partial_+(\epsilon^-\partial_- X)\right)
\end{aligned}
$$

The variation in the action is given by

$$
\begin{aligned}
\delta_\epsilon S = \frac{\mathrm{i}}{2\pi}\sqrt{\frac{2}{\alpha'}}\int \mathrm{d}^2\sigma \Big(&\left(\partial_+(\epsilon^+\psi_+) + \partial_+(\epsilon^-\psi_-)\right)\cdot\partial_- X + \left(\partial_-(\epsilon^+\psi_+) + \partial_-(\epsilon^-\psi_-)\right)\cdot\partial_+ X \\
&- \epsilon^+\partial_+ X \cdot \partial_-\psi_+ - \psi_+ \cdot \partial_-(\epsilon^+\partial_+ X) - \epsilon^-\partial_- X \cdot \partial_+\psi_- - \psi_- \cdot \partial_+(\epsilon^-\partial_- X)\Big)
\end{aligned}
$$

you get $\sim \varepsilon^+ (\partial_+\psi_+ \cdot \partial_- X + \psi_+ \cdot \partial_-\partial_+ X) + \varepsilon^- (\partial_+ X \cdot \partial_-\psi_- + \psi_- \cdot \partial_+\partial_- X)$

$\delta_\varepsilon S = \frac{i}{2\pi}\sqrt{\frac{2}{\alpha'}}\int d^2\sigma \left(\varepsilon^+\partial_+(\psi_+ \cdot \partial_- X) + \varepsilon^-\partial_-(\psi_- \cdot \partial_+ X)\right)$

Boundary : $\psi_+(0) = \psi_-(0)$, $\psi_+(\ell) = \pm\psi_-(\ell)$, $\partial_+ X|_{\sigma=0,\ell} = \partial_- X|_{\sigma=0,\ell}$

$\varepsilon^+(0) = \varepsilon^-(0)$, $\varepsilon^+(\ell) = \pm\varepsilon^-(\ell)$

broken Susy ?!

cancel the total derivative

$$= \frac{\mathrm{i}}{2\pi}\sqrt{\frac{2}{\alpha'}}\int \mathrm{d}^2\sigma \left(-2(\epsilon^+\psi_+ + \epsilon^-\psi_-)\cdot\partial_+\partial_- X - \partial_-(\epsilon^+\partial_+X\cdot\psi_+) - \partial_+(\epsilon^-\partial_-X\cdot\psi_-)\right)$$

local Susy v. global Susy . . .

! [
$$= \frac{\mathrm{i}}{2\pi}\sqrt{\frac{2}{\alpha'}}\int \mathrm{d}^2\sigma \left(-2(\epsilon^+\psi_+ + \epsilon^-\psi_-)\cdot\partial_+\partial_- X\right)$$
$$= 0.$$

if you produce a total derivative, you should check boundary conditions since σ has finite range!

Hence the gauge-fixed action is invariant under the supersymmetry transformations. □

Question 3. Gauge Fixing the RNS-String

The fully covariant, supersymmetric RNS-string is coupled to a world-sheet metric $h^{\alpha\beta}$ and gravitino superpartner χ_α (i.e. what would be called a 2d $N = 1$ supergravity multiplet). Let e_a^α be the zweibein, satisfying

$$e_\alpha^a e_b^\alpha = \delta_b^a, \quad e_a^\alpha e_b^\beta h_{\alpha\beta} = \eta_{ab}$$

The action

$$S^{\text{cov}} = S_B^{\text{cov}} + S_F^{\text{cov}} + S^\chi = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu + \mathrm{i}\overline{\psi}^\mu\gamma^\alpha\partial_\alpha\psi_\mu$$
$$-\frac{\mathrm{i}}{8\pi}\int \mathrm{d}^2\sigma\sqrt{-h}\overline{\chi}_\alpha\gamma^\beta\gamma^\alpha\psi^\mu\left(\sqrt{\frac{2}{\alpha'}}\partial_\beta X_\mu - \frac{\mathrm{i}}{4}\overline{\chi}_\beta\psi_\mu\right)$$

is then invariant under the following supersymmetry transformations

$$\sqrt{\frac{2}{\alpha'}}\delta_\epsilon X^\mu = \mathrm{i}\bar{\epsilon}\psi^\mu$$
$$\delta_\epsilon\psi^\mu = \frac{1}{2}\gamma^\alpha\left(\sqrt{\frac{2}{\alpha'}}\partial_\alpha X^\mu - \frac{\mathrm{i}}{2}\overline{\chi}_\alpha\psi^\mu\right)\epsilon$$
$$\delta_\epsilon e_\alpha^a = \frac{\mathrm{i}}{2}\bar{\epsilon}\gamma^a\chi_\alpha$$
$$\delta_\epsilon\chi_\alpha = 2\nabla_\alpha\epsilon$$

Weyl transformations

$$\delta_\Lambda X^\mu = 0, \quad \delta_\Lambda e_\alpha = \Lambda e_\alpha^a, \quad \delta_\Lambda\psi^\mu = -\frac{1}{2}\Lambda\psi^\mu, \quad \delta_\Lambda\chi_\alpha = \frac{1}{2}\Lambda\chi_\alpha,$$

Super-Weyl transformations

$$\delta_\eta\chi_\alpha = \gamma_\alpha\eta$$

with all others vanishing, 2d Lorentz transformations

$$\delta_\ell X^\mu = 0, \quad \delta_\ell\psi^\mu = -\frac{1}{2}\ell\gamma\psi^\mu, \quad \delta_\ell e_\alpha^a = \ell\varepsilon_b^a e_\alpha^b, \quad \delta_\ell\chi_\alpha = -\frac{1}{2}\ell\gamma_\alpha$$

where $\gamma = \gamma^0\gamma^1$ is the chirality operator, and finally reparametrizations

$$\begin{aligned}
\delta_\xi X^\mu &= -\xi^\beta \partial_\beta X^\mu \\
\delta_\xi \psi^\mu &= -\xi^\beta \partial_\beta \psi^\mu \\
\delta_\xi e^a_\alpha &= -\xi^\beta \partial_\beta e^a_\alpha - e^a_\beta \partial_\alpha \xi^\beta \\
\delta_\xi \chi_\alpha &= -\xi^\beta \partial_\beta \chi_\alpha - \chi_\beta \partial_\alpha \xi^\beta .
\end{aligned}$$

1. Use the bosonic symmetries (two worldsheet reparametrizations ξ, one Lorentz ℓ and one Weyl scaling Λ) to bring the zweibein into the form

$$e^a_\alpha = \delta^a_\alpha$$

   This analysis is very much like in the bosonic string.

2. Use the two supersymmetries and two superconformal symmetries ($\epsilon_\pm$ and $\eta_\pm$) to gauge fix the gravitino to

$$\chi_\alpha = 0.$$

3. Using the equations of motion of e and χ evaluated in the gauged fixing (12) and (13) show that the resulting equations are precisely the Super-Virasoro constraints

$$T_{\pm\pm} = J_\pm = 0.$$

1. Diffeo : $(\sigma, \tau) \mapsto (\tilde\sigma, \tilde\tau)$ s.t. :

$$\tilde h_{00}(\tilde\sigma,\tilde\tau) = -\tilde h_{11}(\tilde\sigma,\tilde\tau) \,, \quad \tilde h_{01}(\tilde\sigma,\tilde\tau) = 0$$

$$\Rightarrow (\tilde h_{\alpha\beta}) = \tilde h_{11}(\tilde\sigma,\tilde\tau)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = e^{2\phi(\tilde\sigma,\tilde\tau)}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Weyl : $\tilde h_{\alpha\beta} \mapsto e^{-2\phi}\tilde h_{\alpha\beta}$

$\Rightarrow \tilde h_{\alpha\beta} = \eta_{\alpha\beta}$ flat metric on worldsheet

$$\tilde h_{\alpha\beta} = \eta_{ab} e^a_\alpha e^b_\beta = \eta_{\alpha\beta} \Rightarrow e^a_\alpha = \delta^a_\alpha$$

2. $\chi_\alpha = \delta^\beta_\alpha \chi_\beta - \frac{1}{2}\gamma_\alpha\gamma^\beta\chi_\beta + \frac{1}{2}\gamma_\alpha\gamma^\beta\chi_\beta$

$$\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta} = \underbrace{\tfrac{1}{2}\gamma_\alpha\gamma^\beta\chi_\beta}_{\hat\chi_\alpha} + \gamma_\alpha \underbrace{\tfrac{1}{2}\gamma^\beta\chi_\beta}_{\lambda}$$

$$\gamma^\alpha \hat\chi_\alpha = 0$$

Susy :

$$\delta_\varepsilon \chi_\alpha = 2\nabla_\alpha \varepsilon = 2\eta^{\alpha\beta}\nabla_\beta\varepsilon = \gamma^\beta\gamma_\alpha\nabla_\beta\varepsilon + \gamma_\alpha\gamma^\beta\nabla_\beta\varepsilon$$

$$(\chi_\alpha + \delta_\varepsilon\chi_\alpha) = (\hat\chi_\alpha + \gamma^\beta\gamma_\alpha\nabla_\beta\varepsilon) + \gamma_\alpha(\lambda + \gamma^\beta\nabla_\beta\varepsilon)$$

$$\Rightarrow \tilde\chi_\alpha = \gamma_\alpha \lambda$$

Super-Weyl : $\tilde\chi_\alpha = 0$

Super-conformal group ? inf-dim ...

Why not starting with the gauge fixed action ?

Not equivalent !! Equation of motion

$$\psi^\mu_\pm \partial_\pm X_\mu = 0 \Rightarrow J_\pm = 0$$