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Problem Sheet 2

C3.11: Riemannian Geometry

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Section A: Introductory

Question 1

Let X, Y be vector fields on (M, g). Let $p \in M$ and let $\alpha : (-\epsilon, \epsilon) \to M$ be the integral curve of X with $\alpha(0) = p$. For all $t \in (-\epsilon, \epsilon)$ let $\tau_t : T_pM \to T_{\alpha(t)}M$ be parallel transport along $\alpha|_{[0,t]}$. Show that

$$\nabla_X Y(p) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\tau_t^{-1} \left(Y\left(\alpha(t)\right)\right)\right)\Big|_{t=0}$$

Proof. We adopt Einstein's convention. A dot always denotes the derivative with respect to t, the parameter of a curve.

Choose a coordinate chart $(U; x^1, ..., x^n)$ at p such that $x^i(p) = 0$. Without loss of generality suppose that $\alpha(-\varepsilon, \varepsilon) \subseteq U$. $\{\partial_1, ..., \partial_n\}$ is a basis of T_pM. We parallelly transport the basis vectors along α. Let $E_i(t) := \tau_t(\partial_i)$. Then $\{E_1, ..., E_n\}$ is a basis of parallel vector fields along α. Suppose that $Y(\alpha(t)) = Y^i(t)E_i(t)$. Since α is the integral curve of X, $X = \dot{\alpha}$ on α. Then

$$\nabla_X Y(p) = \nabla_{\dot{\alpha}} Y(\alpha(t))|_{t=0} = \dot{\alpha}(Y^i(t))E_i(t)|_{t=0} = \dot{Y}^i(0)\partial_i$$

On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\tau_t^{-1} \left(Y\left(\alpha(t)\right)\right)\right)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\tau_t^{-1} \left(Y^i(t)E_i(t)\right)\right)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \left(Y^i(t)\partial_i\right)\Big|_{t=0} = \dot{Y}^i(0)\partial_i$$

The result follows from above. □

Question 2

Let (M, g) be an n-dimensional Riemannian manifold. Let $p \in M$ and let U be a normal neighbourhood of p.

Let $\{E_1, \dots, E_n\}$ be an orthonormal basis for T_pM, let $\psi : T_pM \to \mathbb{R}^n$ be given by $\psi\left(\sum_{i=1}^n x_i E_i\right) = (x_1, \dots, x_n)$ and let $\varphi = \psi \circ \exp_p^{-1} : U \to \mathbb{R}^n$.

(a) Let $\gamma(t)$ be a geodesic through p in U in M. Show that

$$\varphi \circ \gamma(t) = (a_1 t, \dots, a_n t)$$

for $(a_1, \dots, a_n) \in \mathbb{R}^n$.

(b) Show that in (U, φ), we have $g_{ij}(p) = \delta_{ij}$ and $\Gamma^k_{ij}(p) = 0$.

(c) Hence, or otherwise, show that there is open set $V \ni p$ and orthonormal vector fields $E_1, \dots, E_n$ on V such that

$$\nabla_{E_i} E_j(p) = 0$$

Proof. (a)

□

Section B: Core

Question 3

Let (M, g) be a Riemannian manifold. Recall that a **Killing field** on M is a vector field X such that $\mathcal{L}_X g = 0$ or, equivalently, that the flow of X near any point consists of local isometries.

(a) Let $p \in M$ and let U be a normal neighbourhood of p. Suppose that X is a Killing field on (M, g) so that $X(p) = 0$ and $X(q) \neq 0$ for all $q \in U \setminus \{p\}$.

By using the First variation formula, or otherwise, show that X is tangent to all sufficiently small geodesic spheres centred at p.

(b) Show that X is a Killing field on (M, g) if and only if, for all vector fields Y, Z on M,

$$g\left(\nabla_Y X, Z\right) + g\left(\nabla_Z X, Y\right) = 0$$

Proof. A dot always denotes the derivative with respect to t, the parameter of a curve.

(a) Let $\gamma : [0, L] \to U$ be a radial geodesic starting from p. By Proposition 3.15, there exists a variation f of γ such that $X = X_f$ satisfies $X_f(t) = \partial_s f(0, t)$ along γ.The first variation formula is given by

$$\frac{1}{2}E_f'(0) = -\int_0^L g(X_f, \nabla_{\dot\gamma}\dot\gamma)\,\mathrm{d}t + g(X_f(t), \dot\gamma(t))\Big|_{t=0}^{t=L},$$

where the energy[1] is given by

$$E_f(s) = \int_0^L g(\dot f(s,t), \dot f(s,t))\,\mathrm{d}t.$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}s} g(\dot f(s,t), \dot f(s,t))\Big|_{s=0} = (\mathcal{L}_X g)(\dot\gamma(t), \dot\gamma(t)) = 0,$$

since X is a Killing vector field. Hence

$$E_f'(0) = \int_0^L (\mathcal{L}_X g)(\dot\gamma(t), \dot\gamma(t))\,\mathrm{d}t = 0.$$

Since γ is a geodesic, $\nabla_{\dot\gamma}\dot\gamma = 0$. We also have $X_f(0) = X(p) = 0$ by assumption. Substituting into the first variation formula, we have

$$g(X_f(L), \dot\gamma(L)) = 0.$$

X is orthogonal to γ at $t = L$. Since γ is orthogonal to the geodesic spheres centred at p, which have tangent spaces of codimension 1, then X is tangent to the the geodesic spheres centred at p. ✓

(b) For any $Y, Z \in \Gamma(\mathrm{T}M)$, $g(Y, Z) = \mathrm{tr}(g \otimes Y \otimes Z)$, where tr denotes the contraction of *all* covariant and contravariant indices of $g \otimes Y \otimes Z \in \Gamma(\mathrm{T}^2_2 M)$. Since tr commutes with the Lie derivatives, we have

$$\begin{aligned}
\mathcal{L}_X(g(Y,Z)) &= \mathcal{L}_X(\mathrm{tr}(g \otimes Y \otimes Z)) = \mathrm{tr}(\mathcal{L}_X(g \otimes Y \otimes Z)) \\
&= \mathrm{tr}((\mathcal{L}_X g) \otimes Y \otimes Z) + \mathrm{tr}(g \otimes \mathcal{L}_X Y \otimes Z) + \mathrm{tr}(g \otimes Y \otimes \mathcal{L}_X Z) \\
&= (\mathcal{L}_X g)(Y, Z) + g([X, Y], Z) + g(Y, [X, Z]).
\end{aligned}$$

[1] **Cultural Remark**. For a free particule moving along the worldline γ on a Riemannian/Lorentzian manifold M, the Lagrangian is given by $\mathcal{L}[(\gamma(t))] = \frac{1}{2} m g(\dot\gamma, \dot\gamma)$. The energy defined in the lecture should really be called the **action** $S[\gamma] = \int_0^{t_0} \mathcal{L}[\gamma(t)]\,\mathrm{d}t$.

On the other hand, we have

$$\begin{aligned}\mathcal{L}_X(g(Y,Z)) &= X(g(Y,Z)) \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) + g([X,Y], Z) + g(Y, [X,Z]).\end{aligned}$$

Comparing the two equations, we obtain

$$(\mathcal{L}_X g)(Y,Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X)$$

In particular, $\mathcal{L}_X g = 0$ if and only if $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ for all $Y, Z \in \Gamma(\mathrm{T}M)$. □

Perfect.

Question 4

Let (M,g) be a Riemannian manifold, let $f : M \to \mathbb{R}$ be a smooth function and let X be a vector field on M.

(a) Note that we have a linear map from vector fields to vector fields given by $Y \mapsto \nabla_Y X$. We define the **divergence** of X to be the smooth function

$$\operatorname{div} X = \operatorname{tr}(Y \mapsto \nabla_Y X)$$

Show that if X is a Killing field then $\operatorname{div} X = 0$.

(b) Recall that $Y \mapsto g(Y, -)$ defines an isomorphism between vector fields and 1-forms on M. We define the **gradient** of f to be the vector field ∇f given by

$$g(\nabla f, -) = \mathrm{d}f.$$

We define the **Laplacian** of f to be the smooth function

$$\Delta f = \operatorname{div} \nabla f$$

Show that

$$\Delta\left(f^2\right) = 2f\Delta f + 2|\nabla f|^2.$$

Now suppose further that M is compact, connected and oriented with Riemannian volume form Ω.

(c) Show that

$$\mathcal{L}_X \Omega = (\operatorname{div} X)\Omega$$

Relate this to the result about Killing fields from (a).

(d) Show that if $\Delta f \geqslant 0$ on M then f is constant.

Proof. We adopt Einstein's convention. Let us work in coordinates! Let $(U; x^1, \ldots, x^n)$ be a chart and $\{\partial_1, \ldots, \partial_n\}$ be the frame vector fields.

only if the frame is orthonormal at the point p you're considering, which of course you can find

(a) Then $\operatorname{div} X = \nabla_\mu X^\mu = g(\partial_\mu, \nabla_\mu X)$. Since X is a Killing vector field, from Question 3.(b), we have

$$(\mathcal{L}_X g)(\partial_\mu, \partial_\mu) = 2g(\partial_\mu, \nabla_\mu X) = 2 \operatorname{div} X = 0.$$

Hence $\operatorname{div} X = 0$. ✓

(b) The musical isomorphism $\mathrm{T}M \cong \mathrm{T}^*M$ identities the frame vectors ∂_μ with frame covectors $\mathrm{d}x^\mu$. So $\nabla f = \dfrac{\partial f}{\partial x^\mu}\partial_\mu$. Then

$$\nabla(f^2) = \frac{\partial(f^2)}{\partial x^\mu}\partial_\mu = 2f\frac{\partial f}{\partial x^\mu}\partial_\mu = 2f\nabla f$$

And

$$\Delta(f^2) = \mathrm{div}(2f\nabla f) = 2\nabla_\mu(f\nabla f)^\mu = 2\partial_\mu f(\nabla f)^\mu + 2f(\nabla_\mu \nabla f)^\mu = 2g(\nabla f, \nabla f) + 2f\Delta f$$

(c) Let $p \in M$. We take $(U; x^1, ..., x^n)$ to be the geodesic normal coordinates at p, such that

$$x^\mu(p) = 0, \qquad g_{\mu\nu}(p) = \delta_{\mu\nu}, \qquad \Gamma^\lambda_{\mu\nu}(p) = 0.$$

The (pull-back of) volume form at p in the coordinates is given by

$$\Omega = \sqrt{\det(\varphi^* g)}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

By Cartan's formula,

$$\mathcal{L}_X\Omega = \mathrm{d}\circ\iota_X\Omega + \iota_X \circ \mathrm{d}\Omega = \mathrm{d}\circ\iota_X\Omega.$$

If $X = X^\mu\partial_\mu$ at p, then we can compute

$$\begin{aligned}\iota_X\Omega &= \iota_X(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n)\\ &= (-1)^{\mu+1}\iota_X(\mathrm{d}x^\mu)\,\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^\mu} \wedge \cdots \wedge \mathrm{d}x^n\\ &= (-1)^{\mu+1}X^\mu\,\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^\mu} \wedge \cdots \wedge \mathrm{d}x^n.\end{aligned}$$

Then

$$\begin{aligned}\mathrm{d}\circ\iota_X\Omega &= (-1)^{\mu+1}\mathrm{d}X^\mu \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^\mu} \wedge \cdots \wedge \mathrm{d}x^n + (-1)^{\mu+1}X^\mu\,\mathrm{d}\left(\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^\mu} \wedge \cdots \wedge \mathrm{d}x^n\right)\\ &= (-1)^{\mu+1}\partial_\nu X^\mu \mathrm{d}x^\nu \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^\mu} \wedge \cdots \wedge \mathrm{d}x^n\\ &= \partial_\mu X^\mu \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n\\ &= (\mathrm{div}\,X)\Omega.\end{aligned}$$

No Christoffel symbols appeared in the calculation because everything was evaluated at p. Therefore we obtain $\mathcal{L}_X\Omega = (\mathrm{div}\,X)\Omega$ at p. Since p is arbitrary, the relation holds on all M.

(d) *We need to assume that M has no boundary.*

By (c), we can consider the integral

$$\int_M \Delta f\,\Omega = \int_M \mathrm{div}\,\nabla f\,\Omega = \int_M \mathcal{L}_{\nabla f}\Omega = \int_M d\circ i_{\nabla f}\Omega = \oint_{\partial M} i_{\nabla f}\Omega = 0.$$

Since $\Delta f \geqslant 0$ on M, we must have $\Delta f = 0$ on M. By (b) we have

$$\Delta(f^2) = 2g(\nabla f, \nabla f) \geqslant 0,$$

because g is positive definite. Then following the same argument we must have $\Delta(f^2) = 0$ on M. Hence $g(\nabla f, \nabla f) = 0$ on M. By definiteness of g, we have $\nabla f = 0$. Since M is connected, f is constant on M. ☐

Perfect!

Section C: Optional

Question 5

The Euclidean Schwarzschild metric (of mass $m > 0$) is defined for $\left(\cos \frac{t}{4m}, \sin \frac{t}{4m}\right) \in \mathcal{S}^1$, $r > 2m, \theta \in (0, \pi)$ and $(\cos\phi, \sin\phi) \in \mathcal{S}^1$ by

$$g = \left(1 - \frac{2m}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \,\mathrm{d}\theta^2 + r^2 \sin^2\theta \,\mathrm{d}\phi^2$$

and extends smoothly to $\theta = 0, \pi$.

(a) Show that there are no geodesics in this metric with r constant.

(b) Show that, given any point p with $r > 2m$ there exists a finite length geodesic γ starting at p ending at a point q with $r = 2m$.

Proof. (a) We can derive the radial equation for a geodesic exactly the same way as in General Relativity (where the Schwarzschild metric is Lorentzian).

The Lagrangian is given by

$$\mathcal{L}[\gamma(s)] = \left(1 - \frac{2m}{r}\right) \dot{t}^2 + \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 + r^2 \left(\dot{\theta}^2 + \sin^2\theta \dot{\varphi}^2\right),$$

where the dot denotes the derivative with respect to the affine parameter s. We observe that t and φ are ignorable coordinates. We have

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = 2\left(1 - \frac{2m}{r}\right) \dot{t} = \text{const}, \qquad \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = 2r^2 \sin^2\theta \dot{\varphi} = \text{const}.$$

We can use the SO(3) symmetry of the manifold to fix the geodesics on the plane $\theta = \pi/2$. Then $\dot{\theta} = 0$. We set the constants $J := r^2\dot{\varphi}$ and $E := \left(1 - \frac{2m}{r}\right)\dot{t}$, which are the angular momentum and energy per unit mass.

Since γ is affinely parametrised, we have $\mathcal{L}[\gamma(s)] = g(\dot{\gamma}, \dot{\gamma}) = 1$. This gives

$$\mathcal{L} = \left(1 - \frac{2m}{r}\right) \dot{t}^2 + \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 + r^2\dot{\varphi}^2 = \left(1 - \frac{2m}{r}\right)^{-1} E^2 + \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 + \frac{J^2}{r^2} = 1.$$

Suppose that there is a geodesic with $r = \text{const}$. Then

$$E^2 = \left(1 - \frac{J^2}{r^2}\right)\left(1 - \frac{2m}{r}\right)$$

□